LETKF

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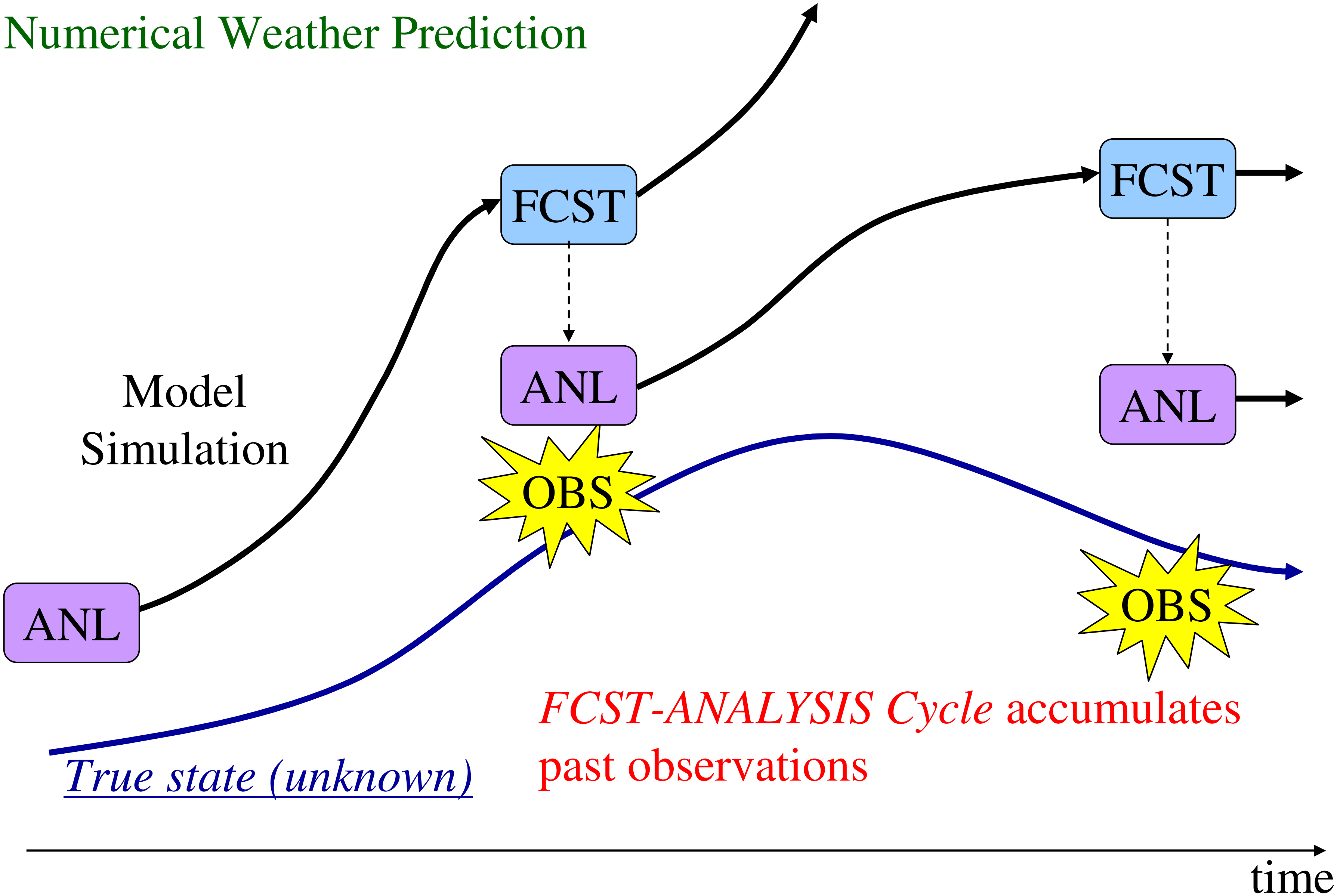
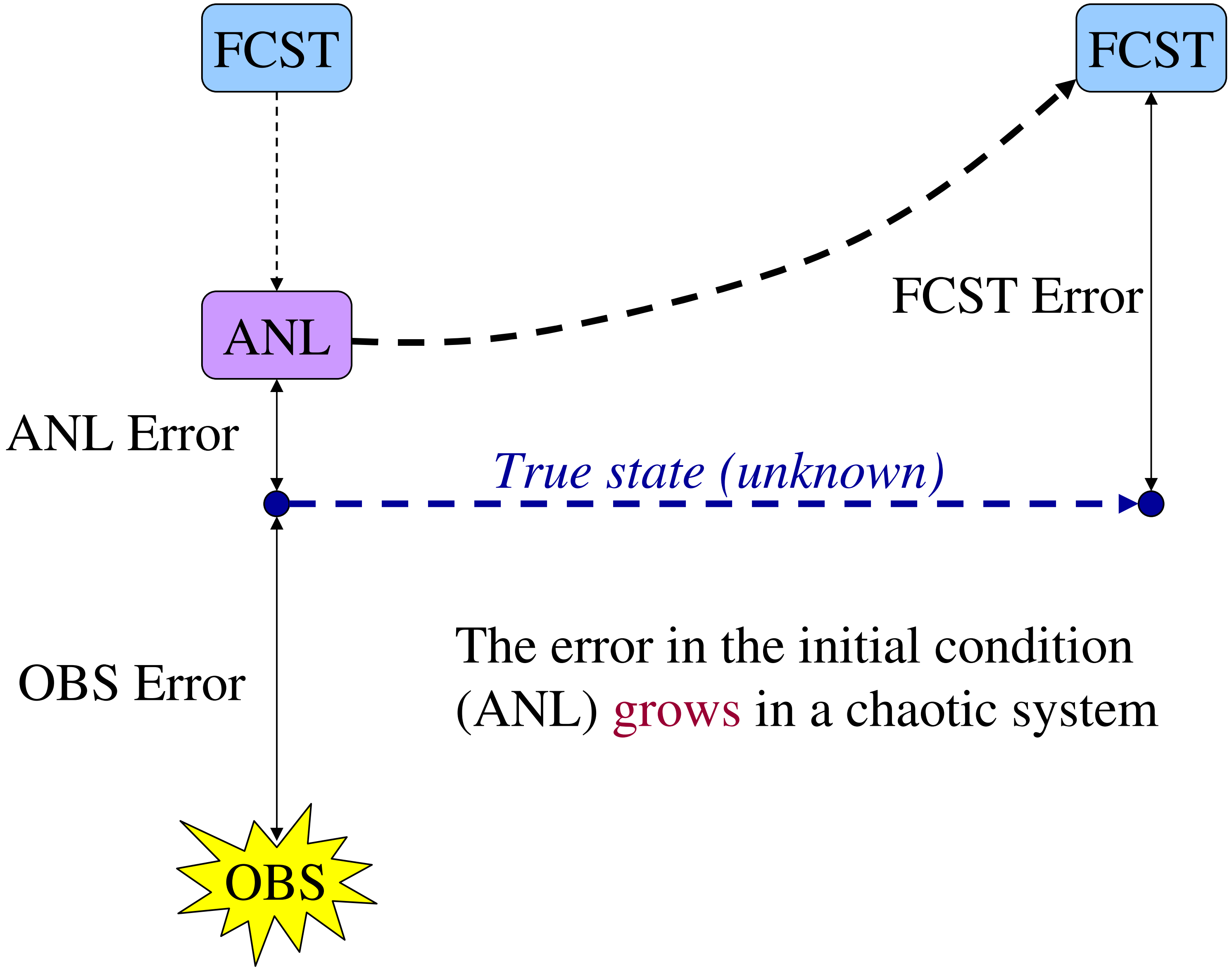
Data assimilation in NWP

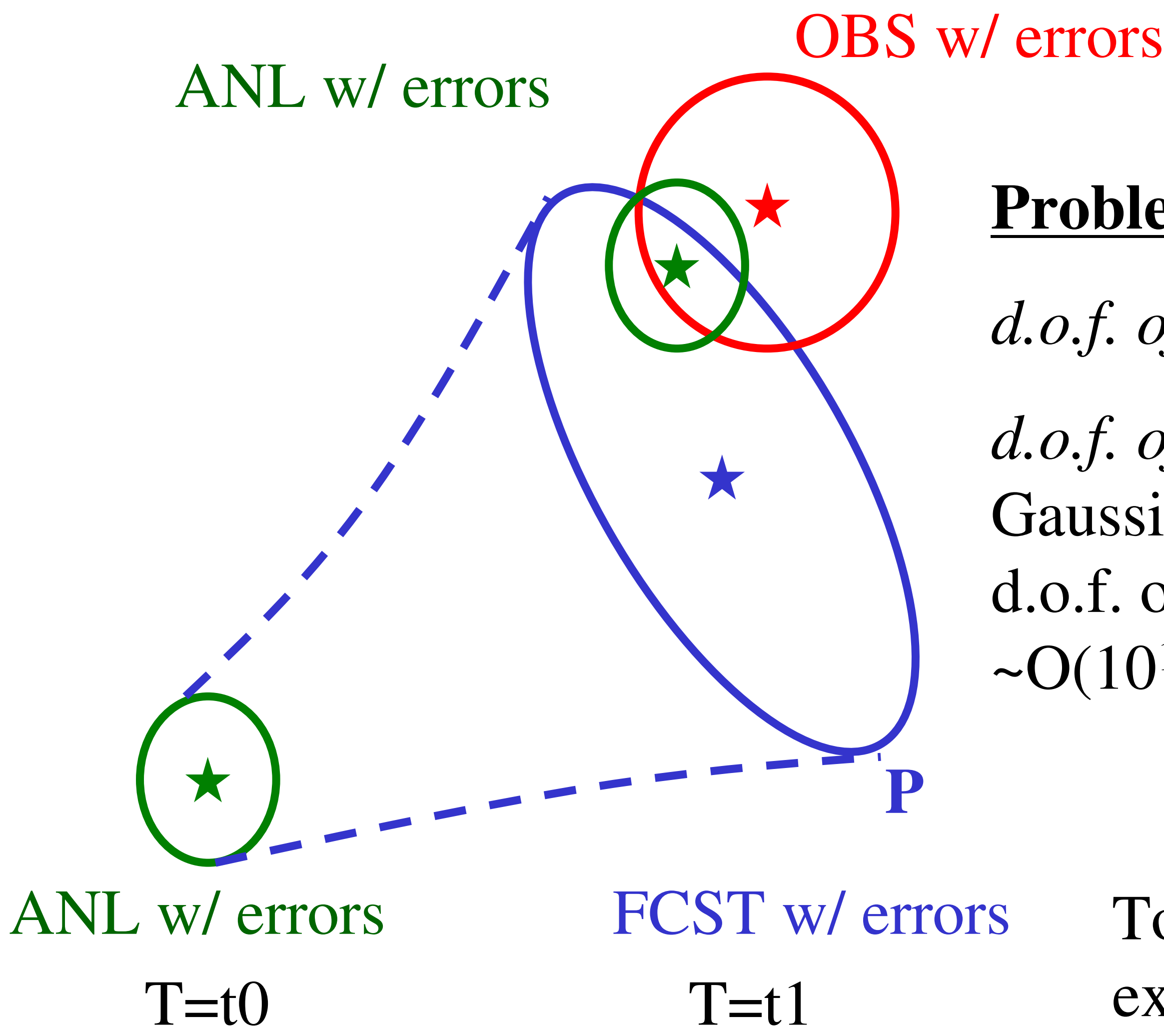


Data Assimilation = Analysis



The error in the initial condition (ANL) grows in a chaotic system

Probabilistic view



Problem :

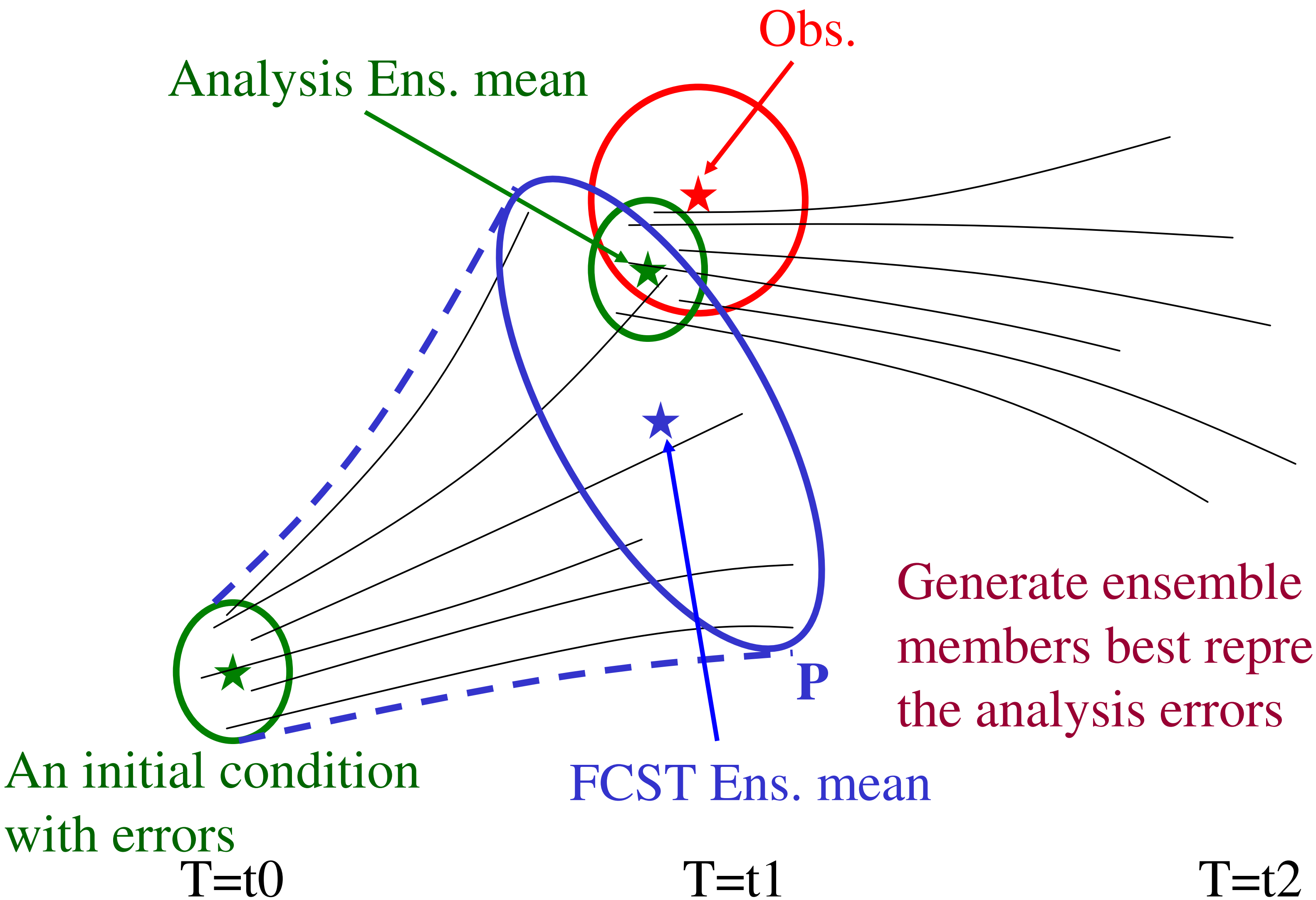
d.o.f. of the system: ~O(10^6)

d.o.f. of the error: even Gaussian distribution has d.o.f. of the covariance ~O(10^{12})

↓

Too large to express explicitly

A schematic of EnKF

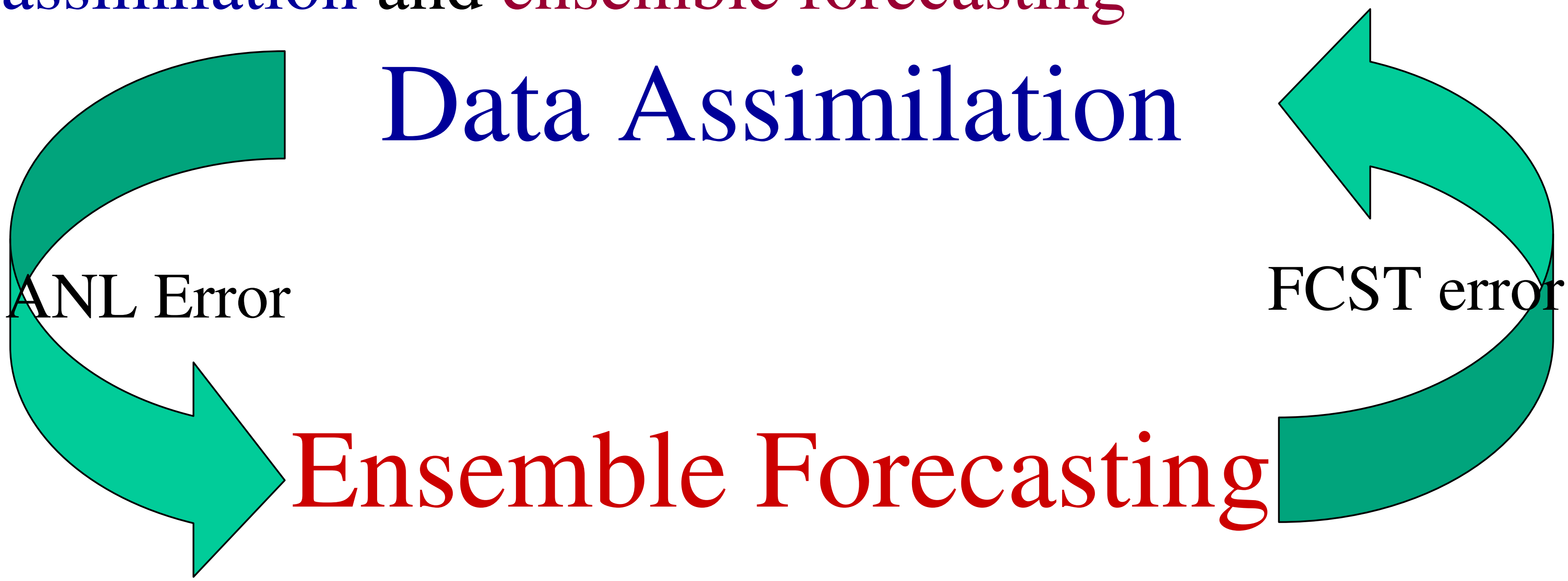


Generate ensemble members best representing the analysis errors

EnKF = ensemble fcst. + ensemble update

A core concept of EnKF

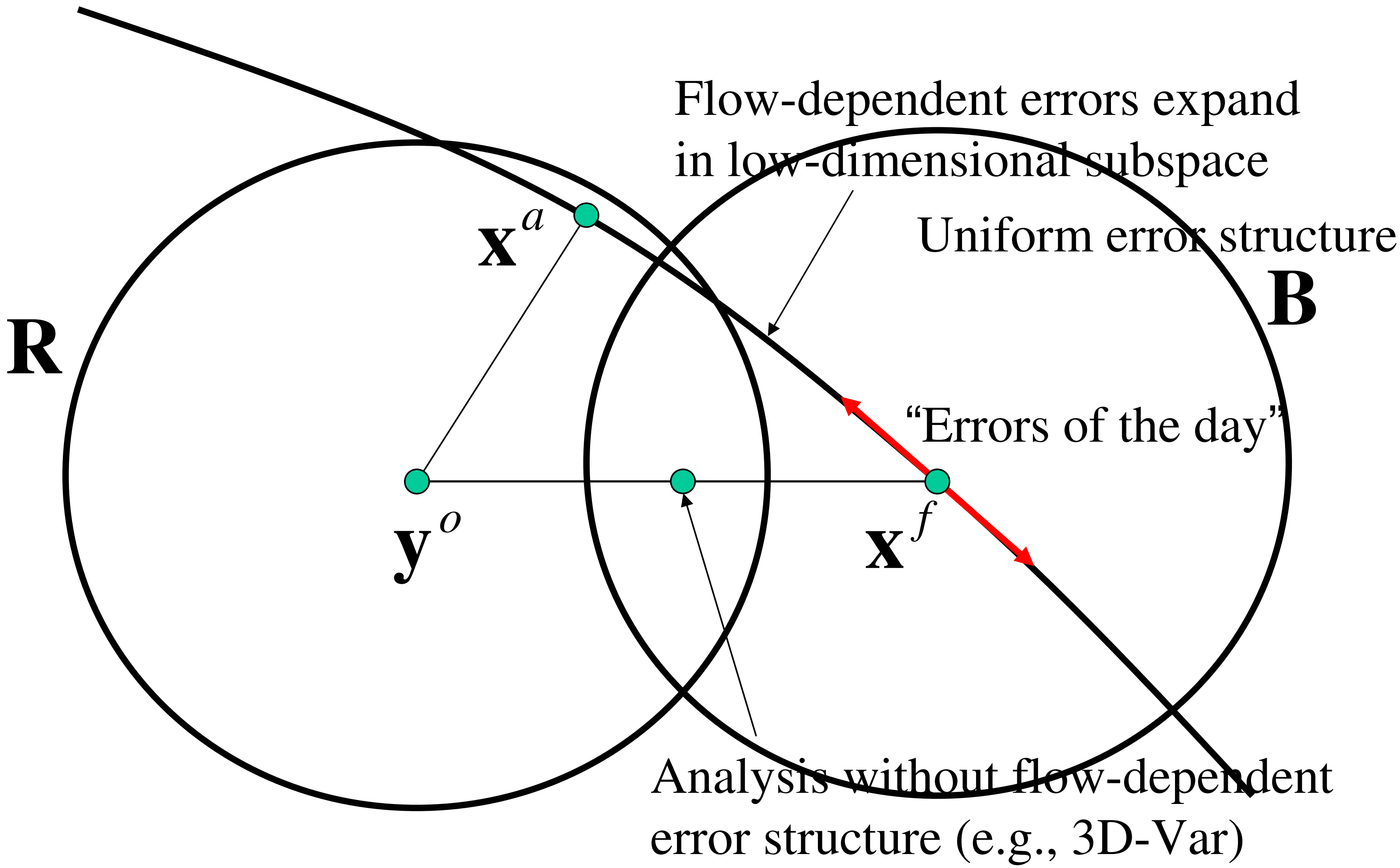
Complementary relationship between data assimilation and ensemble forecasting



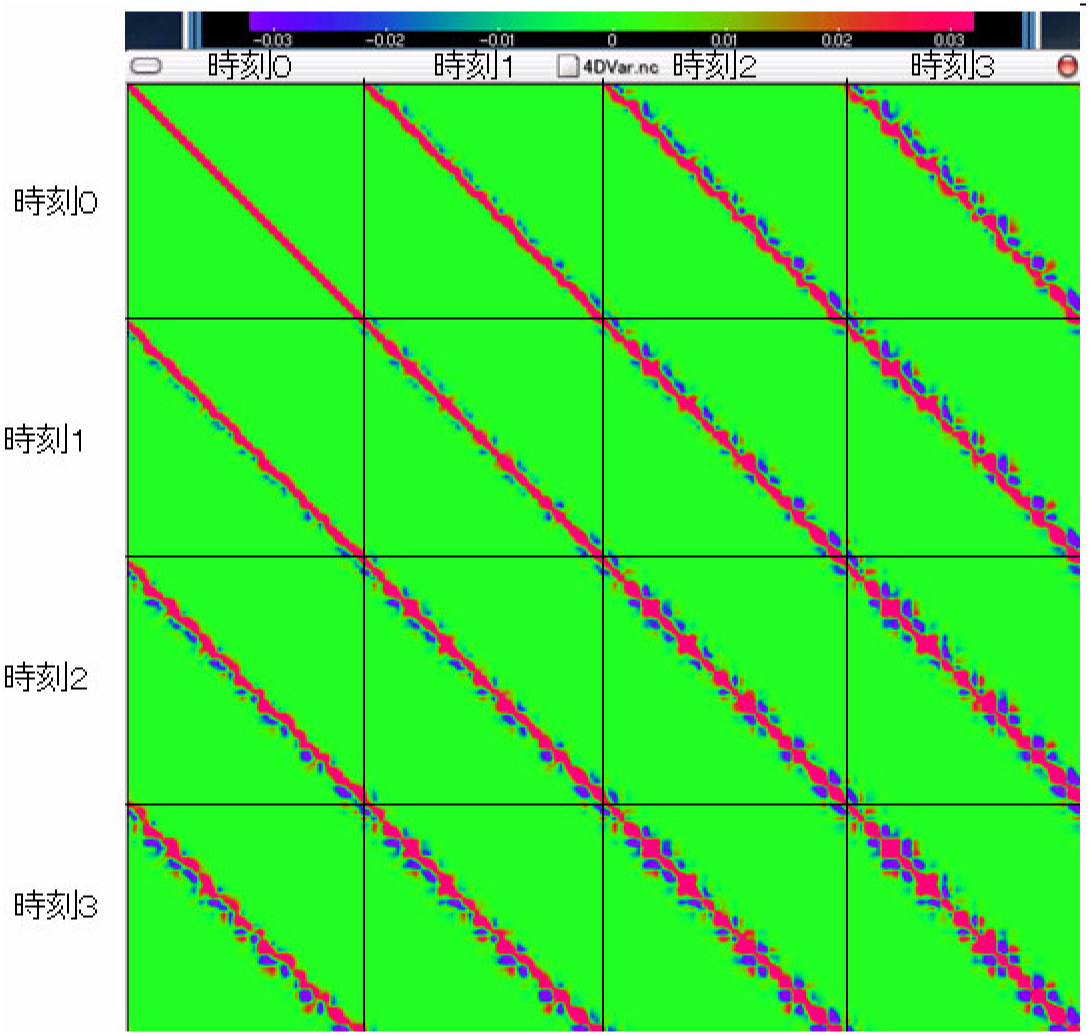
This cycle process = EnKF

Analyze with the flow-dependent forecast error, ensemble forecast with initial ensemble reflecting the analysis error

Difference between EnKF and 3D-Var



Flow-dependence of B in L96



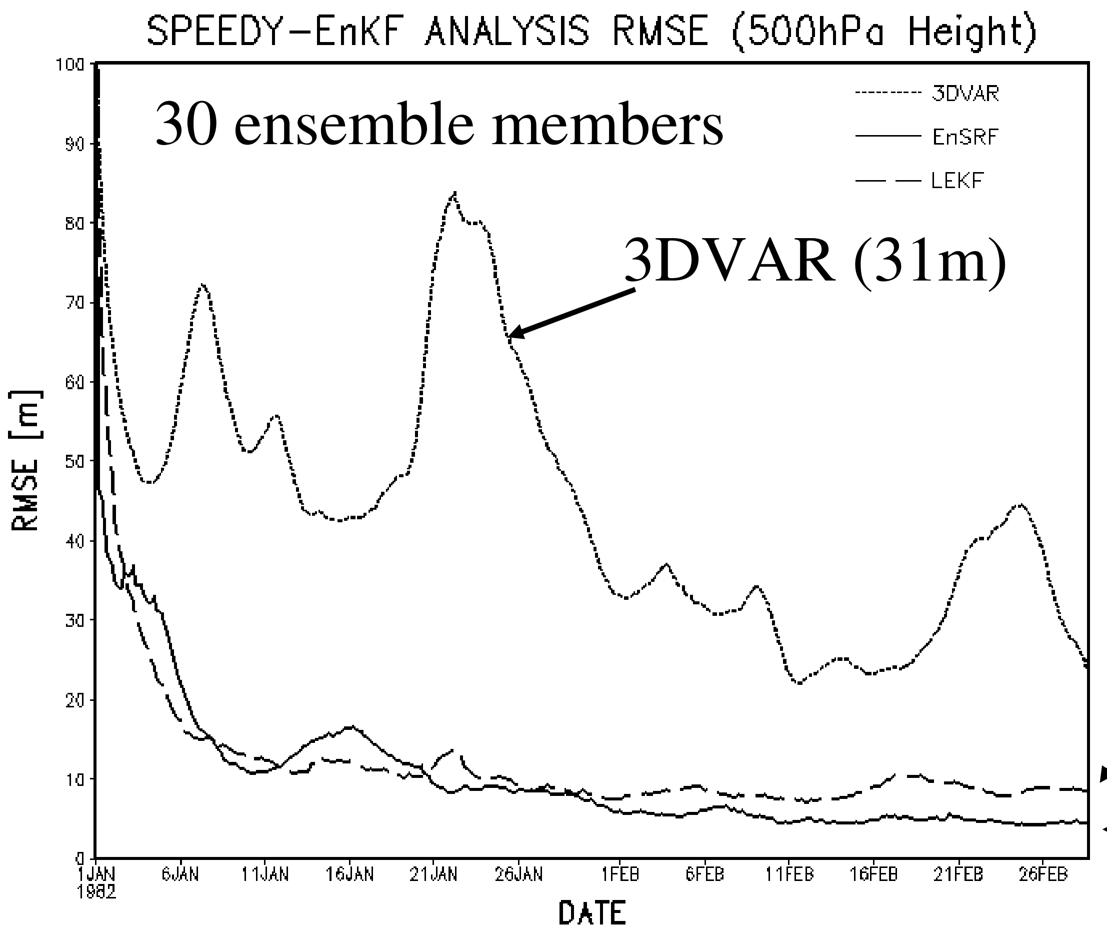
Courtesy of Hotta

Error covariance matrix of the Lorenz-96 system with different time levels, showing time evolution of B

An example of EnKF analysis accuracy

EnKF is advantageous to traditional data assimilation methods including 3D-Var, currently in operations at several NWP centers.

Many centers (ECMWF, JMA, UK MetOffice, MeteorFrance, Canada, etc.) switched to 4D-Var which also considers flow-dependent error structures.



EnKF vs. 4D-Var

	EnKF	4D-Var
“advanced” method?	Y	Y
Simple to code?	Y	N (e.g., Minimizer)
Adjoint model?	N	Y
Observation operator	Only forward (e.g., TC center)	Adjoint required
Asynchronous obs?	Y (4D-EnKF)	Y (intrinsic)
Initialization after analysis?	N	Y
Analysis errors?	Y (ensemble ptb)	N
Limitation	ensemble size	Assim. window
	EnKF with infinite ensemble size and 4D-Var with infinite window are equivalent.	

EnKF - summary

- EnKF considers flow-dependent error structures, or the "errors of the day"
  - "advanced" data assimilation method
    - 4D-Var is also an "advanced" method. How different?

- EnKF analyzes the analysis errors in addition to analysis itself
  - "ideal" ensemble perturbations

LETKF - Local Ensemble Transform Kalman Filter

- Invented by the Chaos Group (e.g., Profs. Eugenia Kalnay, Ed Ott, Jim Yorke, Brian Hunt, and more) at the University of Maryland
- Selected References of the method:
  - LEKF by Ott et al. (2004, Tellus)
  - LETKF by Hunt et al. (2007, Physica D)
- More references and information:
  - http://www.weatherchaos.umd.edu/

KF and EnKF

Kalman Filter	Ensemble Kalman Filter
$\mathbf{x}:[N] \quad \mathbf{P}:[N\times N]$	$\mathbf{X}:[N\times m]$
Forecast equations	*Ensemble forecasts*
$\mathbf{x}_i^f = M(\mathbf{x}_{i-1}^a)$	$\mathbf{X}_i^f = [M(\mathbf{x}_{i-1}^{a(1)}) \mid \cdots \mid M(\mathbf{x}_{i-1}^{a(m)})] \equiv M(\mathbf{X}_{i-1}^a)$
$\mathbf{P}_i^f = \mathbf{M}_{\mathbf{x}_{i-1}^a}\mathbf{P}_{i-1}^a\mathbf{M}_{\mathbf{x}_{i-1}^a}{}^T + \mathbf{Q}$	Approximated by $\mathbf{P}_i^f \approx \dfrac{\delta\mathbf{X}_i^f(\delta\mathbf{X}_i^f)^T}{m-1}$
Kalman gain	$\delta\mathbf{Y} = H(\delta\mathbf{X}):[p\times m]$
$\mathbf{K}_i = \mathbf{P}_i^f\mathbf{H}^T[\mathbf{H}\mathbf{P}_i^f\mathbf{H}^T + \mathbf{R}]^{-1}$	$\mathbf{K}_i = \delta\mathbf{X}_i^f(\delta\mathbf{Y})^T[\underbrace{\delta\mathbf{Y}(\delta\mathbf{Y})^T + (m-1)\mathbf{R}}_{\text{[pxp] matrix inverse}}]^{-1}$
Analysis equations	
$\mathbf{x}_i^a = \mathbf{x}_i^f + \mathbf{K}_i(\mathbf{y}_i^o - H(\mathbf{x}_i^f))$	Solve for the ensemble mean
$\mathbf{P}_i^a = [\mathbf{I} - \mathbf{K}_i\mathbf{H}]\mathbf{P}_i^f$	Ensemble perturbations $\delta\mathbf{X}^a = [(m-1)\mathbf{P}^a]^{1/2}$

LETKF (Hunt 2005; Hunt et al. 2007; Ott et al. 2004)

- Two categories of the EnKF (Ensemble Kalman Filter)

Perturbed observation (PO) method	Square root filter (SRF)
Classical	Relatively new
Already in operations (Canadian EPS)	Not in operations yet
Additional sampling errors by PO	No such additional sampling errors

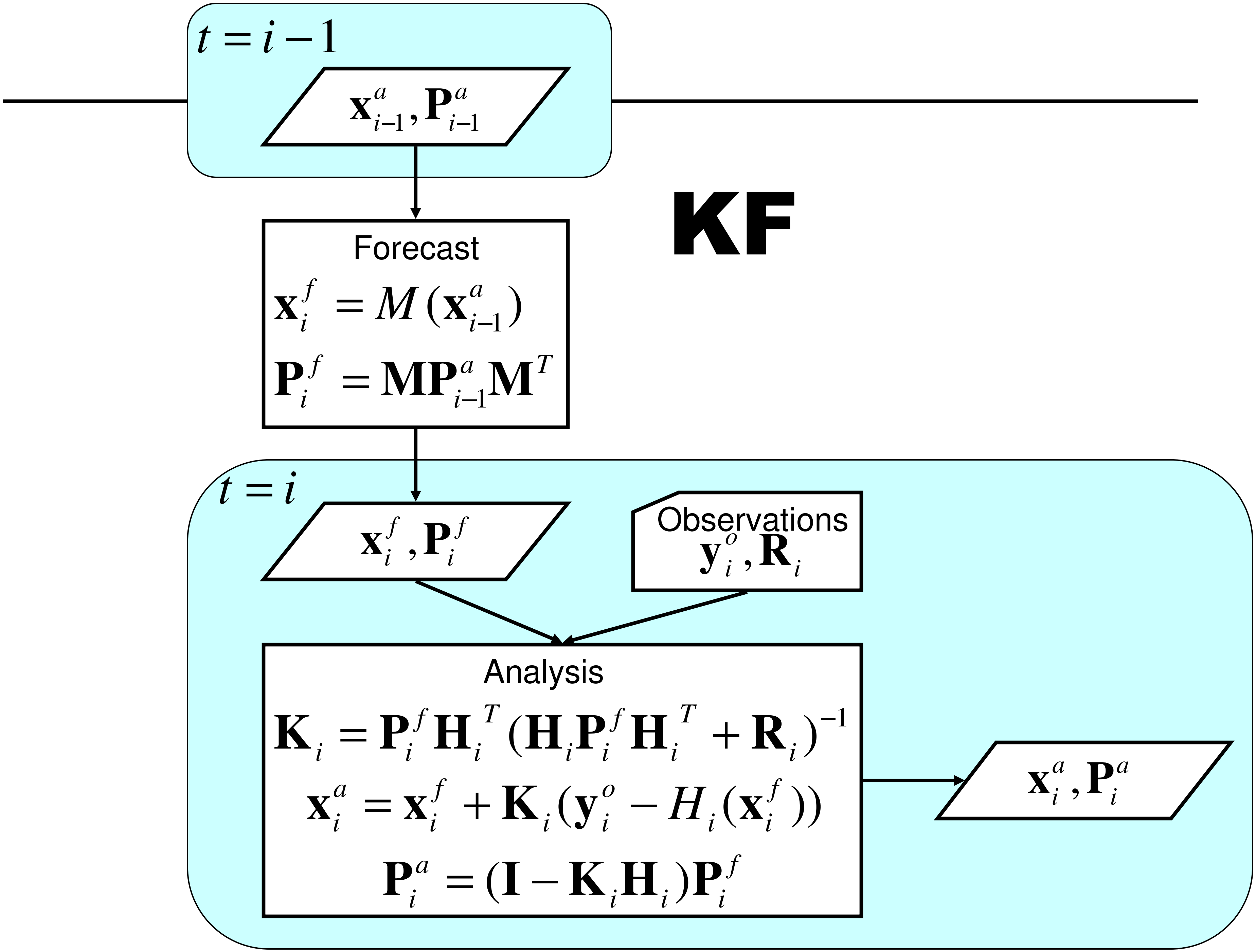
- LETKF (Local Ensemble Transform Kalman Filter)
  - is a kind of ensemble square root filter (SRF)
  - is efficient with the parallel architecture

PO vs SRF

$$\delta \mathbf{X}_i^a = \delta \mathbf{X}_i^f \mathbf{T}$$

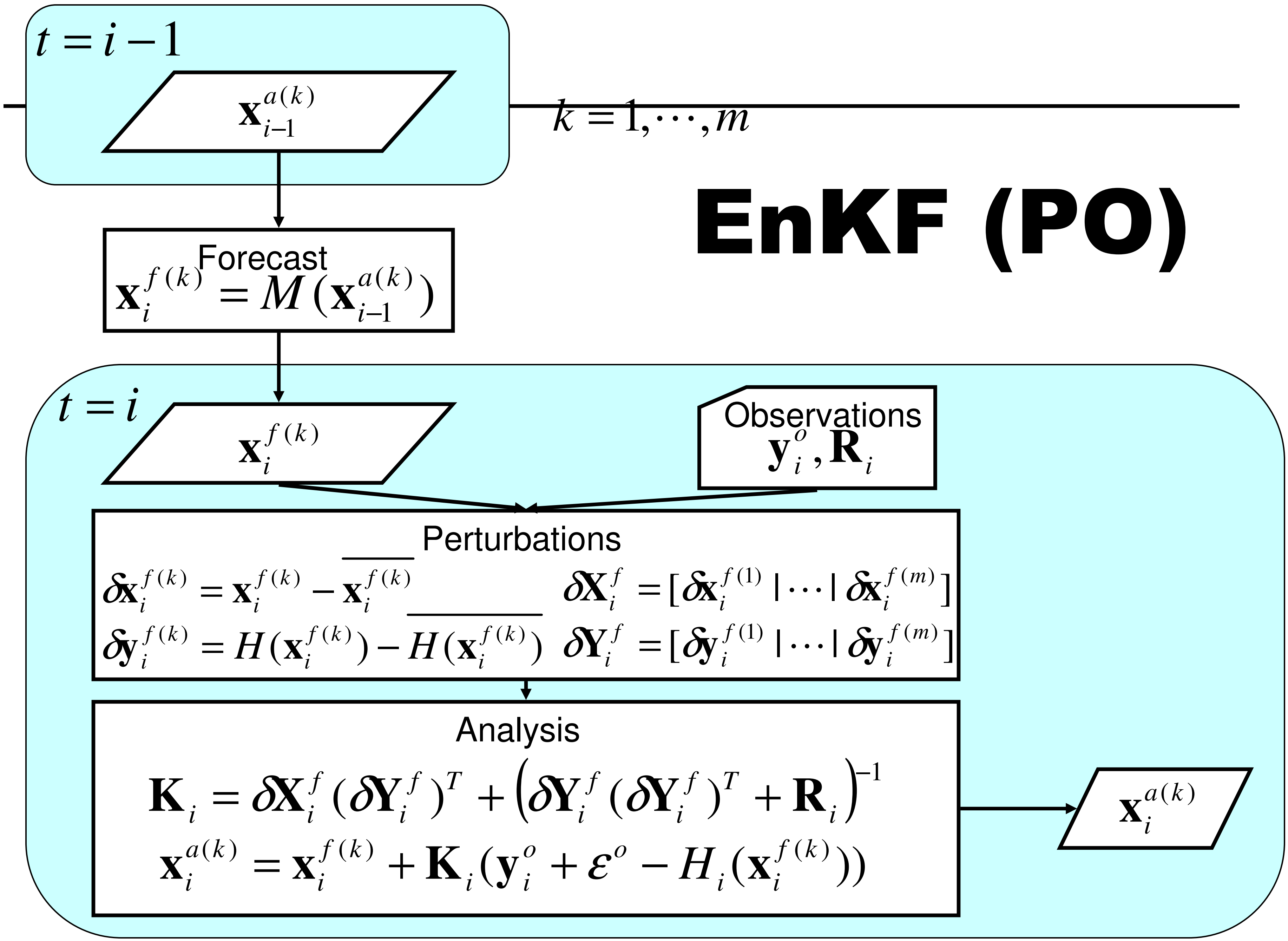
$$\delta \mathbf{X}^f \mathbf{T}\mathbf{T}^T (\delta \mathbf{X}^f)^T = (\mathbf{I} - \mathbf{K}\mathbf{H}) \delta \mathbf{X}^f (\delta \mathbf{X}^f)^T$$

KF



$t = i - 1$
$\mathbf{x}^a_{i-1}, \mathbf{P}^a_{i-1}$
Forecast
$\mathbf{x}^f_i = M(\mathbf{x}^a_{i-1})$
$\mathbf{P}^f_i = \mathbf{M}\mathbf{P}^a_{i-1}\mathbf{M}^T$
$t = i$
$\mathbf{x}^f_i, \mathbf{P}^f_i$
Observations
$\mathbf{y}^o_i, \mathbf{R}_i$
Analysis
$\mathbf{K}_i = \mathbf{P}^f_i \mathbf{H}_i^T (\mathbf{H}_i \mathbf{P}^f_i \mathbf{H}_i^T + \mathbf{R}_i)^{-1}$
$\mathbf{x}^a_i = \mathbf{x}^f_i + \mathbf{K}_i(\mathbf{y}^o_i - H_i(\mathbf{x}^f_i))$
$\mathbf{P}^a_i = (\mathbf{I} - \mathbf{K}_i \mathbf{H}_i)\mathbf{P}^f_i$
$\mathbf{x}^a_i, \mathbf{P}^a_i$

EnKF (PO)



$t = i-1$
$\mathbf{x}_{i-1}^{a(k)}$
$k = 1, \cdots, m$
Forecast
$\mathbf{x}_i^{f(k)} = M(\mathbf{x}_{i-1}^{a(k)})$
$t = i$
$\mathbf{x}_i^{f(k)}$
Observations
$\mathbf{y}_i^o, \mathbf{R}_i$
Perturbations
$\delta\mathbf{x}_i^{f(k)} = \mathbf{x}_i^{f(k)} - \overline{\mathbf{x}_i^{f(k)}}$
$\delta\mathbf{X}_i^f = [\delta\mathbf{x}_i^{f(1)} | \cdots | \delta\mathbf{x}_i^{f(m)}]$
$\delta\mathbf{y}_i^{f(k)} = H(\mathbf{x}_i^{f(k)}) - \overline{H(\mathbf{x}_i^{f(k)})}$
$\delta\mathbf{Y}_i^f = [\delta\mathbf{y}_i^{f(1)} | \cdots | \delta\mathbf{y}_i^{f(m)}]$
Analysis
$\mathbf{K}_i = \delta\mathbf{X}_i^f (\delta\mathbf{Y}_i^f)^T + \left(\delta\mathbf{Y}_i^f (\delta\mathbf{Y}_i^f)^T + \mathbf{R}_i\right)^{-1}$
$\mathbf{x}_i^{a(k)} = \mathbf{x}_i^{f(k)} + \mathbf{K}_i(\mathbf{y}_i^o + \varepsilon^o - H_i(\mathbf{x}_i^{f(k)}))$
$\mathbf{x}_i^{a(k)}$

Why do we need to perturb obs?

$$\mathbf{x}_i^{a(k)} = \mathbf{x}_i^{f(k)} + \mathbf{K}_i(\mathbf{y}_i^o - H_i(\mathbf{x}_i^{f(k)}))$$

$$\delta\mathbf{x}_i^{a(k)} \approx \delta\mathbf{x}_i^{f(k)} - \mathbf{K}_i\mathbf{H}_i\delta\mathbf{x}_i^{f(k)}$$

$$\overline{\mathbf{x}}_i^a = \frac{1}{m}\sum_{k=1}^{m}\mathbf{x}_i^{a(k)}$$

$$\delta X^a = (I - KH)\delta X^f$$

$$\mathbf{P}_i^a = \delta\mathbf{X}_i^a(\delta\mathbf{X}_i^a)^T$$

$$= (\mathbf{I} - \mathbf{K}_i\mathbf{H}_i)\mathbf{P}_i^f(\mathbf{I} - \mathbf{K}_i\mathbf{H}_i)^T$$

$$\mathbf{P}_i^a = \left\langle \varepsilon_i^a(\varepsilon_i^a)^T \right\rangle$$

$$= \left\langle \left((\mathbf{I} - \mathbf{K}_i\mathbf{H}_i)\varepsilon_i^f + \mathbf{K}_i\varepsilon_i^o\right)\left((\mathbf{I} - \mathbf{K}_i\mathbf{H}_i)\varepsilon_i^f + \mathbf{K}_i\varepsilon_i^o\right)^T \right\rangle$$

$$= (\mathbf{I} - \mathbf{K}_i\mathbf{H}_i)\mathbf{P}_i^f(\mathbf{I} - \mathbf{K}_i\mathbf{H}_i)^T + \boxed{\mathbf{K}_i\mathbf{R}_i\mathbf{K}_i^T} + cross$$

LETKF algorithm (Hunt, 2005, et al., 2007)

$$\mathbf{P}^f \approx \frac{\delta\mathbf{X}^f (\delta\mathbf{X}^f)^T}{m-1} = \delta\mathbf{X}^f \underline{\tilde{\mathbf{P}}^f} (\delta\mathbf{X}^f)^T \qquad \tilde{\mathbf{P}}^f = \frac{\mathbf{I}}{m-1} : [m \times m]$$

In the space spanned by $\delta\mathbf{X}^f$

$$\tilde{\mathbf{P}}^a = [\underline{(m-1)\mathbf{I}/\rho + (\delta\mathbf{Y})^T \mathbf{R}^{-1} \delta\mathbf{Y}}]^{-1} = \mathbf{U}\mathbf{D}^{-1}\mathbf{U}^T$$

Eigenvalue decomposition: $\mathbf{U}\mathbf{D}\mathbf{U}^T : [m \times m]$

Analysis equations

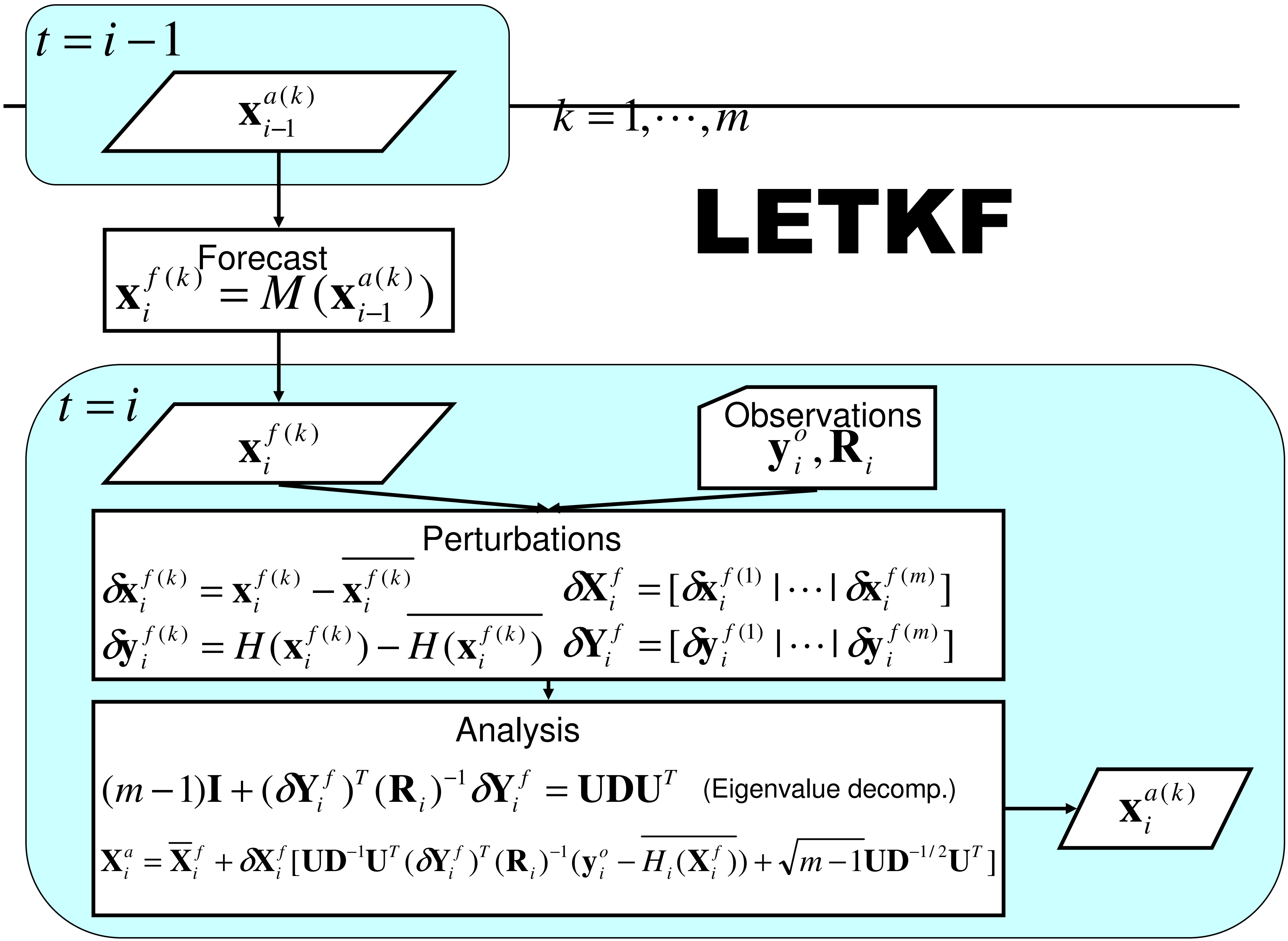
$$\overline{\mathbf{x}}^a = \overline{\mathbf{x}}^f + \delta\mathbf{X}^f \tilde{\mathbf{P}}^a (\delta\mathbf{Y})^T \mathbf{R}^{-1} (\mathbf{y}^o - \overline{H(\mathbf{x}^f)})$$

$$\delta\mathbf{X}^a = \delta\mathbf{X}^f [(m-1)\tilde{\mathbf{P}}^a]^{1/2} = \delta\mathbf{X}^f \sqrt{m-1}\,\mathbf{U}\mathbf{D}^{-1/2}\mathbf{U}^T$$

LETKF analysis

$$\mathbf{X}^a = \overline{\mathbf{x}}^f + \underbrace{\delta\mathbf{X}^f \left(\tilde{\mathbf{P}}^a (\delta\mathbf{Y})^T \mathbf{R}^{-1} (\mathbf{y}^o - \overline{H(\mathbf{x}^f)}) + \sqrt{m-1}\,\mathbf{U}\mathbf{D}^{-1/2}\mathbf{U}^T \right)}_{\text{Ensemble analysis increments}}$$

LETKF



$t = i - 1$
$\mathbf{x}_{i-1}^{a(k)}$
$k = 1, \cdots, m$
Forecast
$\mathbf{x}_i^{f(k)} = M(\mathbf{x}_{i-1}^{a(k)})$
$t = i$
$\mathbf{x}_i^{f(k)}$
Observations
$\mathbf{y}_i^o, \mathbf{R}_i$
Perturbations
$\delta\mathbf{x}_i^{f(k)} = \mathbf{x}_i^{f(k)} - \overline{\mathbf{x}_i^{f(k)}}$
$\delta\mathbf{X}_i^f = [\delta\mathbf{x}_i^{f(1)} | \cdots | \delta\mathbf{x}_i^{f(m)}]$
$\delta\mathbf{y}_i^{f(k)} = H(\mathbf{x}_i^{f(k)}) - \overline{H(\mathbf{x}_i^{f(k)})}$
$\delta\mathbf{Y}_i^f = [\delta\mathbf{y}_i^{f(1)} | \cdots | \delta\mathbf{y}_i^{f(m)}]$
Analysis
$(m-1)\mathbf{I} + (\delta\mathbf{Y}_i^f)^T (\mathbf{R}_i)^{-1} \delta\mathbf{Y}_i^f = \mathbf{UDU}^T$
(Eigenvalue decomp.)
$\mathbf{X}_i^a = \overline{\mathbf{X}}_i^f + \delta\mathbf{X}_i^f [\mathbf{UD}^{-1}\mathbf{U}^T (\delta\mathbf{Y}_i^f)^T (\mathbf{R}_i)^{-1} (\mathbf{y}_i^o - \overline{H_i(\mathbf{X}_i^f)}) + \sqrt{m-1}\,\mathbf{UD}^{-1/2}\mathbf{U}^T]$
$\mathbf{x}_i^{a(k)}$

Derivation of LETKF eqs.

$$(\mathbf{I}-\mathbf{K}_i\mathbf{H}_i)\delta\mathbf{X}_i^f(\delta\mathbf{X}_i^f)^T$$
$$=\delta\mathbf{X}_i^f(m-1)[(m-1)\mathbf{I}+(\mathbf{H}_i\delta\mathbf{X}_i^f)^T(\mathbf{R}_i)^{-1}\mathbf{H}_i\delta\mathbf{X}_i^f]^{-1}(\delta\mathbf{X}_i^f)^T$$

$$\mathbf{T}\mathbf{T}^T=(m-1)[\underline{(m-1)\mathbf{I}+(\mathbf{H}_i\delta\mathbf{X}_i^f)^T(\mathbf{R}_i)^{-1}\mathbf{H}_i\delta\mathbf{X}_i^f}]^{-1}$$

Eigenvalue decomposition: $\mathbf{U}\mathbf{D}\mathbf{U}^T : [m\times m]$

$$\mathbf{T}=\sqrt{m-1}\,\mathbf{U}\mathbf{D}^{-1/2}\mathbf{U}^T$$

$$\mathbf{K}_i=\mathbf{P}_i^a\mathbf{H}_i^T\mathbf{R}_i^{-1}$$

$$\mathbf{K}_i=\delta\mathbf{X}_i^f\mathbf{U}\mathbf{D}^{-1}\mathbf{U}^T(\mathbf{H}_i\delta\mathbf{X}_i^f)^T(\mathbf{R}_i)^{-1}$$

$$\mathbf{X}_i^a=\bar{\mathbf{X}}_i^f+\mathbf{K}_i(\mathbf{y}_i^o-\overline{H_i(\mathbf{X}_i^f)})+\delta\mathbf{X}_i^a$$
$$=\bar{\mathbf{X}}_i^f+\delta\mathbf{X}_i^f[\mathbf{U}\mathbf{D}^{-1}\mathbf{U}^T(\mathbf{H}_i\delta\mathbf{X}_i^f)^T(\mathbf{R}_i)^{-1}(\mathbf{y}_i^o-\overline{H_i(\mathbf{X}_i^f)})+\sqrt{m-1}\,\mathbf{U}\mathbf{D}^{-1/2}\mathbf{U}^T]$$