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Variational assimilation has been shown to yield significant improvements in the quality of numerical forecasts.

It has also been invaluable for re-analysis:
The ERA-40 Project at ECMWF was carried out using the 3D-Var system.

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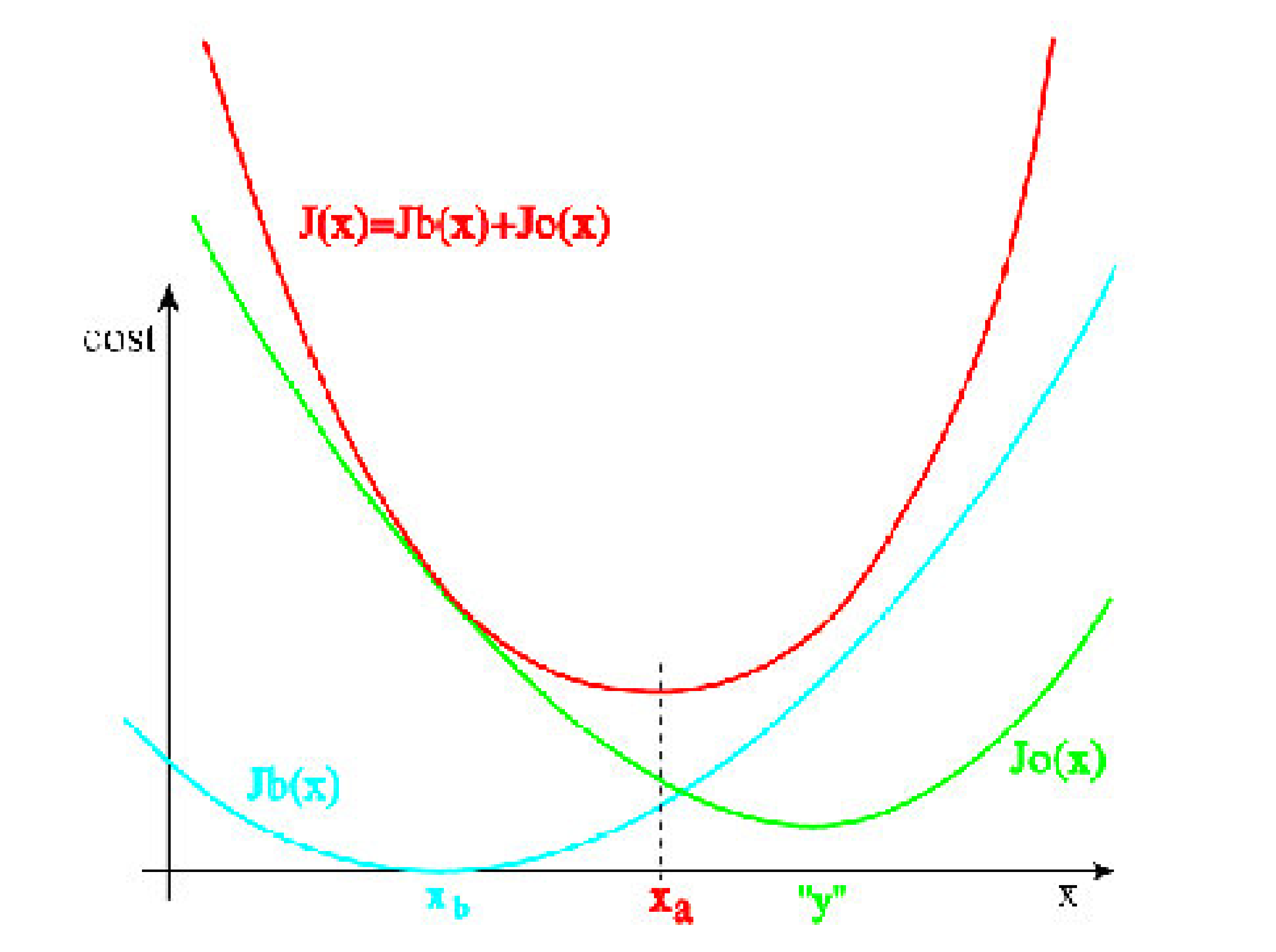
The same equivalence holds for the full 3-dimensional case.

Lorenc (1986) showed that the OI solution is equivalent to a specific variational assimilation problem: Find the optimal analysis $\mathbf{x}_a$ field that minimizes a (scalar) cost function.

The cost function is defined as the (weighted) distance between $\mathbf{x}$ and the background $\mathbf{x}_b$, *plus the (weighted) distance to the observations* $\mathbf{y}_{o,}$:

$$J(\mathbf{x}) = \frac{1}{2}\left\{(\mathbf{x}-\mathbf{x}_b)^T\mathbf{B}^{-1}(\mathbf{x}-\mathbf{x}_b) + [\mathbf{y}_o - H(\mathbf{x})]^T\mathbf{R}^{-1}[\mathbf{y}_o - H(\mathbf{x})]\right\}$$



Schematic representation of the cost function in a simple one-dimensional case. J_b and J_o respectively tend to pull the analysis towards the background x_b and the observation y.

Again, the cost function is

$$J(\mathbf{x}) = \frac{1}{2}\left\{(\mathbf{x}-\mathbf{x}_b)^T\mathbf{B}^{-1}(\mathbf{x}-\mathbf{x}_b) + [\mathbf{y}_o - H(\mathbf{x})]^T\mathbf{R}^{-1}[\mathbf{y}_o - H(\mathbf{x})]\right\}$$

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Then we can linearize the observation operator:

$$[\mathbf{y}_o - H(\mathbf{x})] = \mathbf{y}_o - H[\mathbf{x}_b + (\mathbf{x}-\mathbf{x}_b)] = \{\mathbf{y}_o - H(\mathbf{x}_b)\} - \mathbf{H}\cdot(\mathbf{x}-\mathbf{x}_b)$$

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We now substitute this into the cost function:

$$J(\mathbf{x}) = \tfrac{1}{2}\left\{(\mathbf{x}-\mathbf{x}_b)^T\mathbf{B}^{-1}(\mathbf{x}-\mathbf{x}_b) + [\mathbf{y}_o - H(\mathbf{x})]^T\mathbf{R}^{-1}[\mathbf{y}_o - H(\mathbf{x})]\right\}$$

The result is:

$$2J(\mathbf{x}) = (\mathbf{x}-\mathbf{x}_b)^T\mathbf{B}^{-1}(\mathbf{x}-\mathbf{x}_b)$$
$$+\big[\{\mathbf{y}_o - H(\mathbf{x}_b)\} - \mathbf{H}(\mathbf{x}-\mathbf{x}_b)\big]^T\mathbf{R}^{-1}\big[\{\mathbf{y}_o - H(\mathbf{x}_b)\} - \mathbf{H}(\mathbf{x}-\mathbf{x}_b)\big]$$

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Expanding the products, we get

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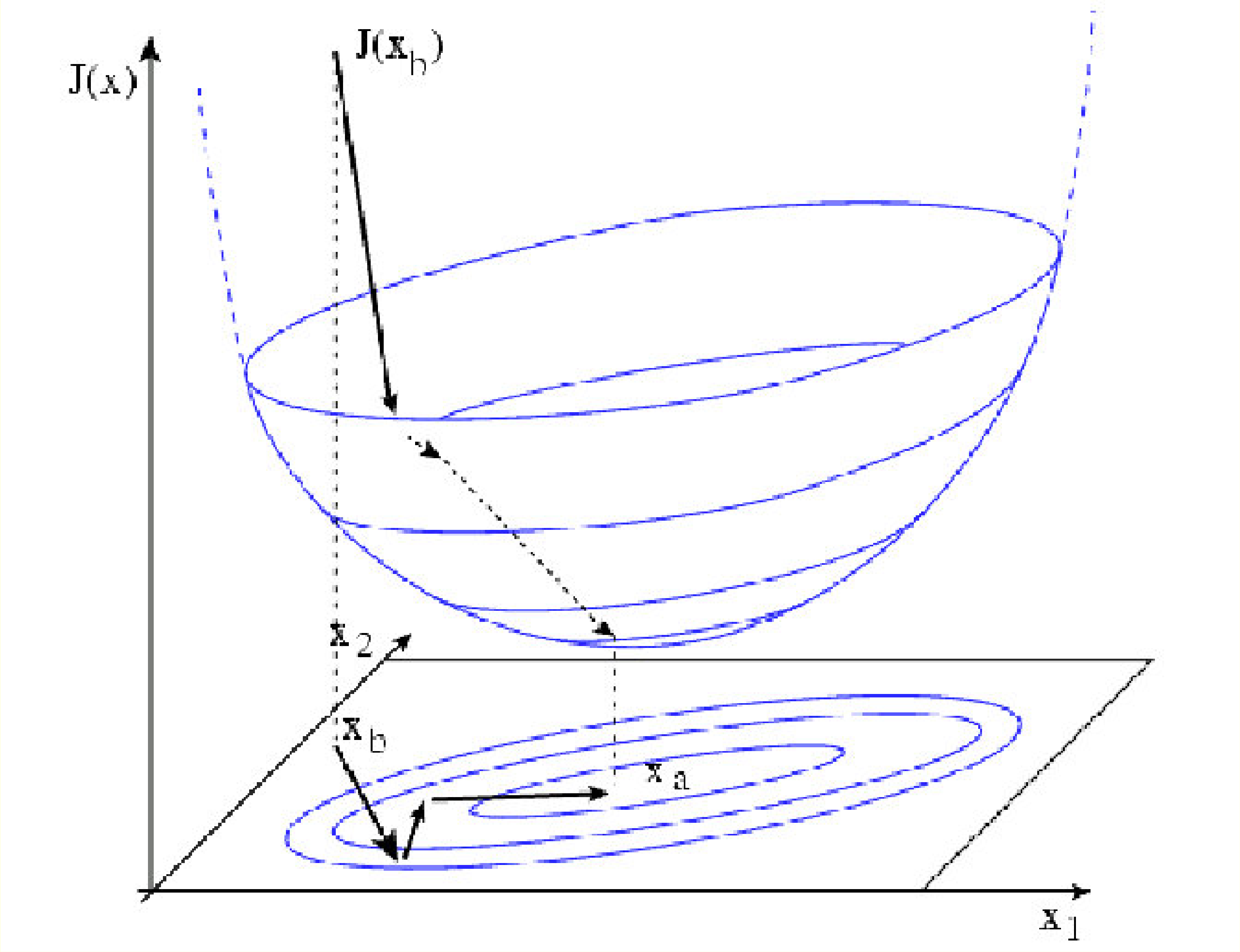
The cost function is a quadratic function of the analysis increments $(\mathbf{x}-\mathbf{x}_b)$.



Schematic of the cost function in two dimensions. The minimum is found by moving down-gradient in discrete steps.

We need to compute the gradient of J.

We use the following Lemma:

Given a quadratic function $F(\mathbf{x}) = \frac{1}{2}\mathbf{x}^T\mathbf{A}\mathbf{x} + \mathbf{d}^T\mathbf{x} + c$***, where*** $\mathbf{A}$ ***is a symmetric matrix,*** $\mathbf{d}$ ***is a vector and*** c ***a scalar, the gradient is given by*** $\nabla F(\mathbf{x}) = \mathbf{A}\mathbf{x} + \mathbf{d}$***.***

⋆ ⋆ ⋆

Proof:

$$F(\mathbf{x}) = \frac{1}{2}\sum_i \sum_j A_{ij}x_i x_j + \sum_i d_i x_i + c$$

So the derivative w.r.t. x_k **is**

$$\frac{\partial F}{\partial x_k} = \frac{1}{2}\sum_j A_{kj}x_j + \frac{1}{2}\sum_j A_{ik}x_i + d_k = \sum_j A_{kj}x_j + d_k$$

Q.E.D.

Recall the cost function was

$$\begin{aligned}
2J(\mathbf{x}) &= (\mathbf{x}-\mathbf{x}_b)^T\mathbf{B}^{-1}(\mathbf{x}-\mathbf{x}_b) + (\mathbf{x}-\mathbf{x}_b)^T\mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}(\mathbf{x}-\mathbf{x}_b) \\
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Combining the first two terms, we get

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2J(\mathbf{x}) &= (\mathbf{x}-\mathbf{x}_b)^T[\mathbf{B}^{-1} + \mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}](\mathbf{x}-\mathbf{x}_b) \\
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The gradient of the cost function J with respect to $\mathbf{x}$ is

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\nabla J(\mathbf{x}) = [\mathbf{B}^{-1} + \mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}](\mathbf{x}-\mathbf{x}_b) - \mathbf{H}^T\mathbf{R}^{-1}\{\mathbf{y}_o - H(\mathbf{x}_b)\}
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Repeat: The gradient of J with respect to $\mathbf{x}$ is

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We now set $\nabla J(\mathbf{x}_a) = 0$ to ensure that J is a minimum, and obtain an equation for $(\mathbf{x}_a - \mathbf{x}_b)$

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The equivalence is *not obvious*.

Simple demo of equivalence of 3D-Var and OI

$$(\mathbf{B}^{-1} + \mathbf{H}^T\mathbf{R}^{-1}\mathbf{H})^{-1}\mathbf{H}^T\mathbf{R}^{-1} = ? = (\mathbf{B}\mathbf{H}^T)(\mathbf{R} + \mathbf{H}\mathbf{B}\mathbf{H}^T)^{-1}$$

$$\mathbf{H}^T\mathbf{R}^{-1}(\mathbf{R} + \mathbf{H}\mathbf{B}\mathbf{H}^T) = ? = (\mathbf{B}^{-1} + \mathbf{H}^T\mathbf{R}^{-1}\mathbf{H})(\mathbf{B}\mathbf{H}^T)$$

$$\mathbf{H}^T + \mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\mathbf{B}\mathbf{H}^T = ? = \mathbf{H}^T + \mathbf{H}^T\mathbf{R}^{-1}\mathbf{H}\mathbf{B}\mathbf{H}^T$$

Yeah!!!

Again, the variational analysis is

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The equivalence between the minimization of the analysis error variance and the three-dimensional variational cost function approach is an important property.

Conclusion of the foregoing

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For a selection of techniques, see Numerical Recipes, which may be inspected online before purchase.

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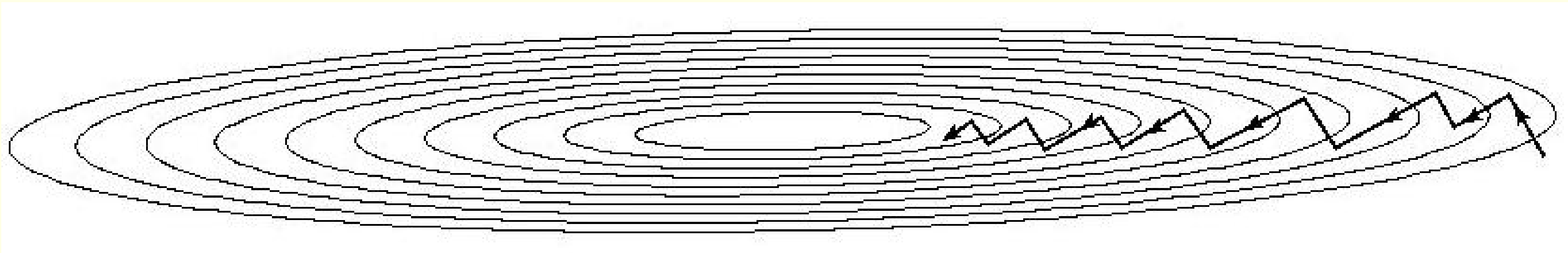

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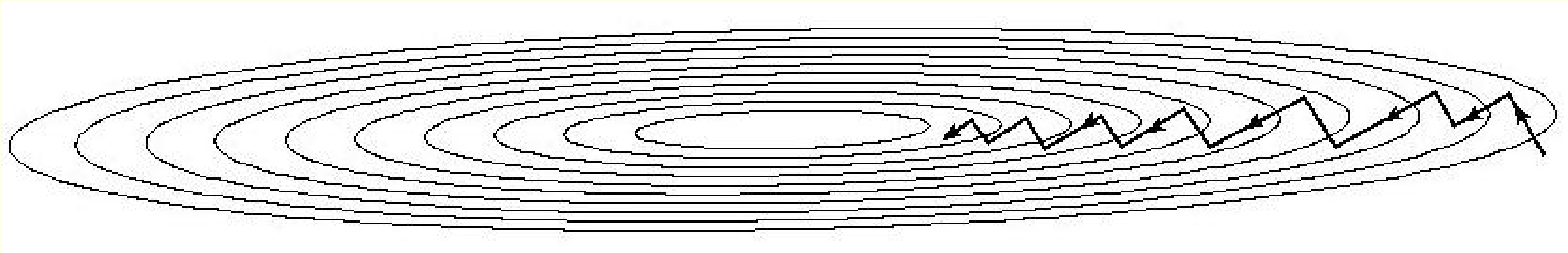

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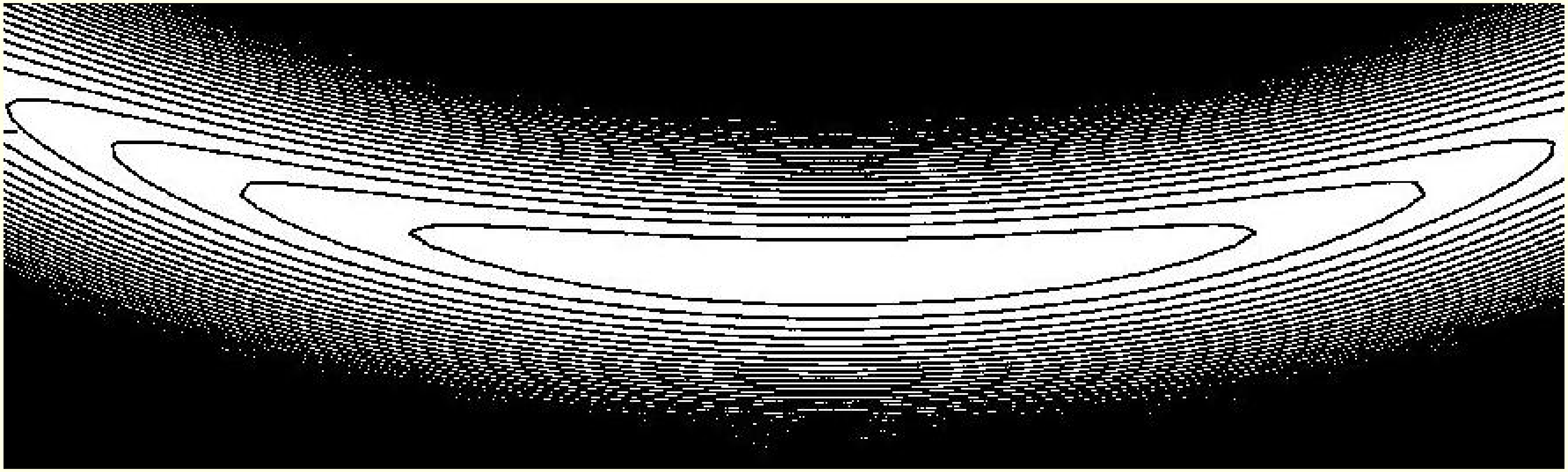
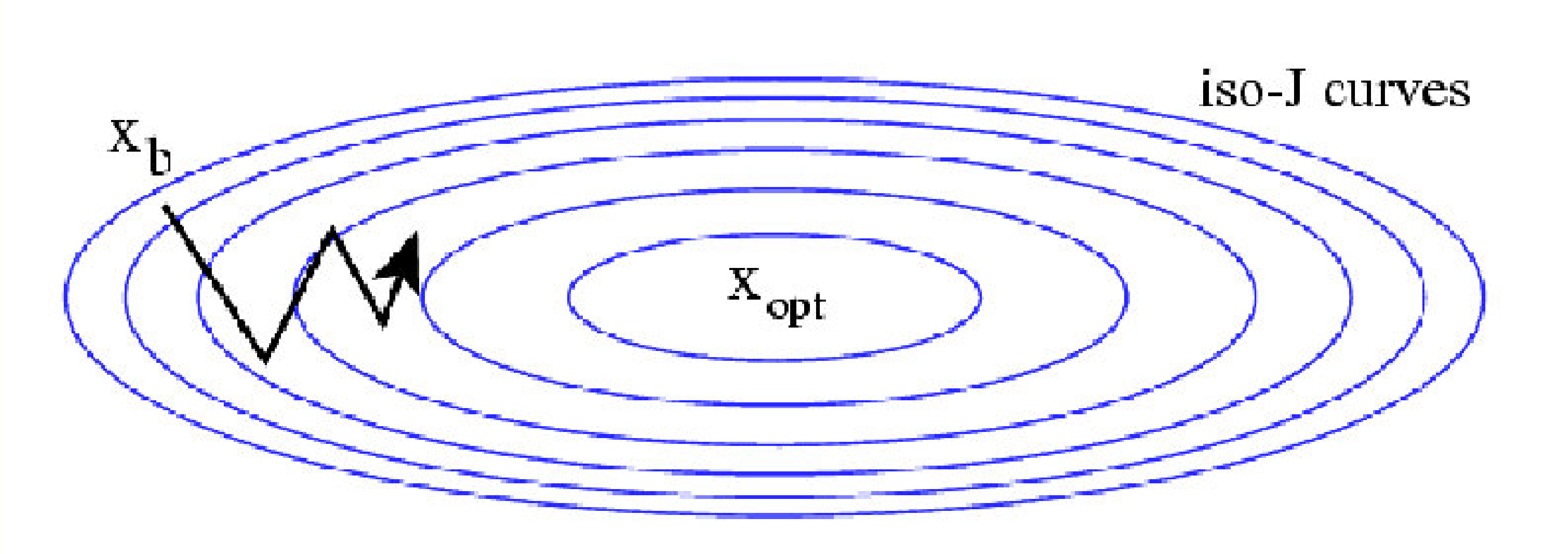
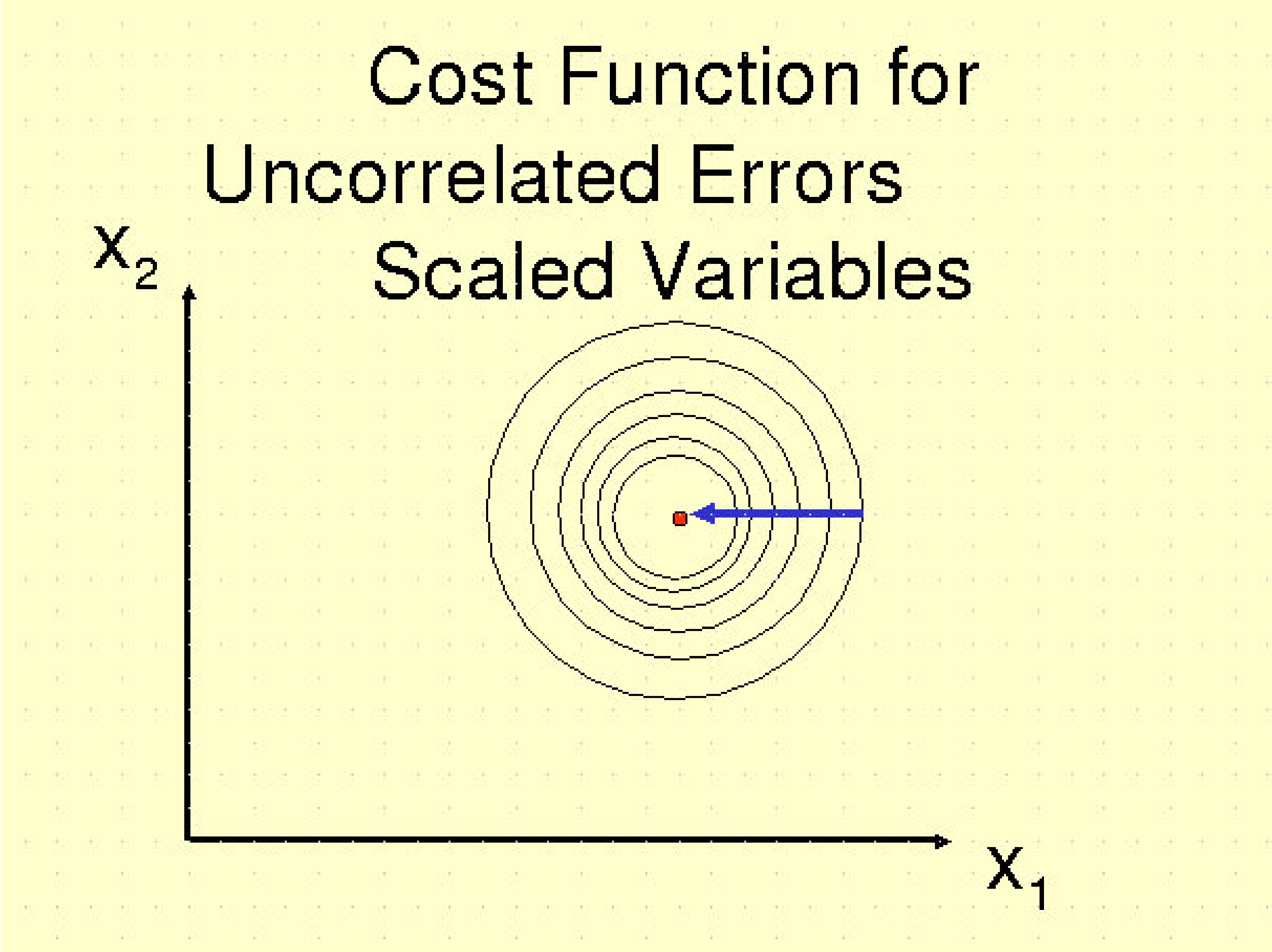


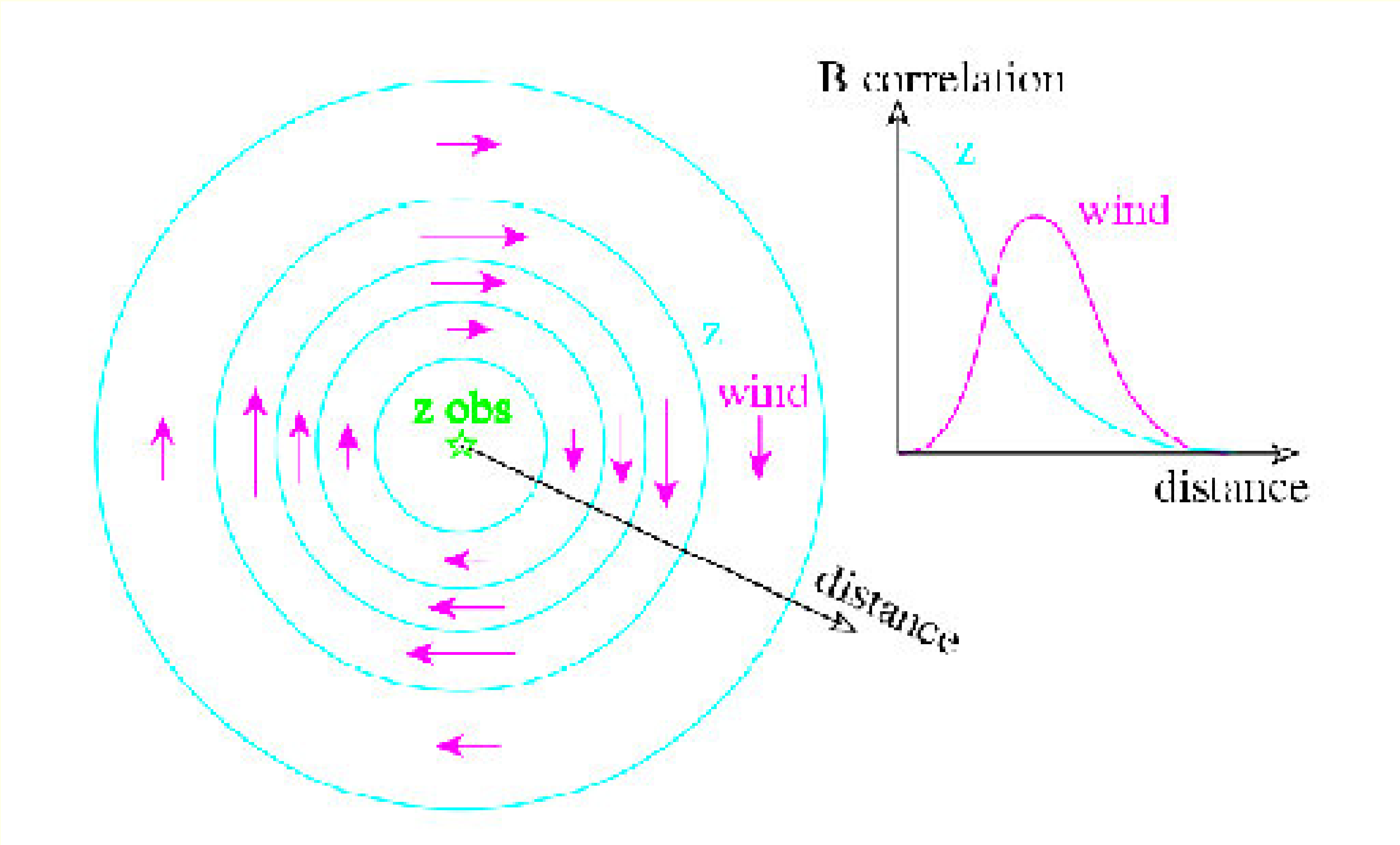
For a banana shaped surface, the minimum is much harder to find.



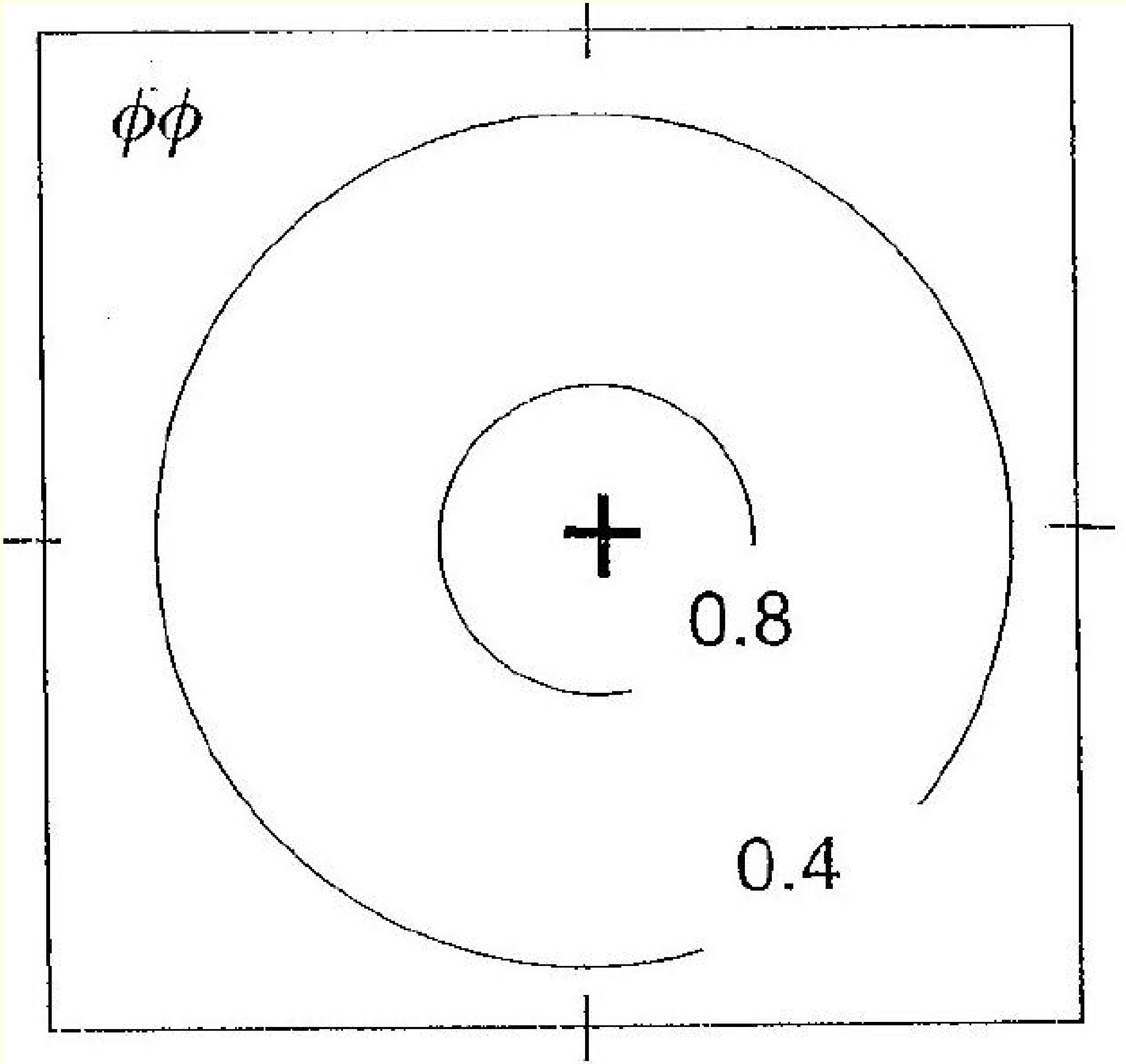
The "narrow-valley" effect. The minimization can spend many iterations zigzagging towards the minimum.


Cost Function for
Uncorrelated Errors
Scaled Variables
x_2
x_1

J_B: The Conventional Method



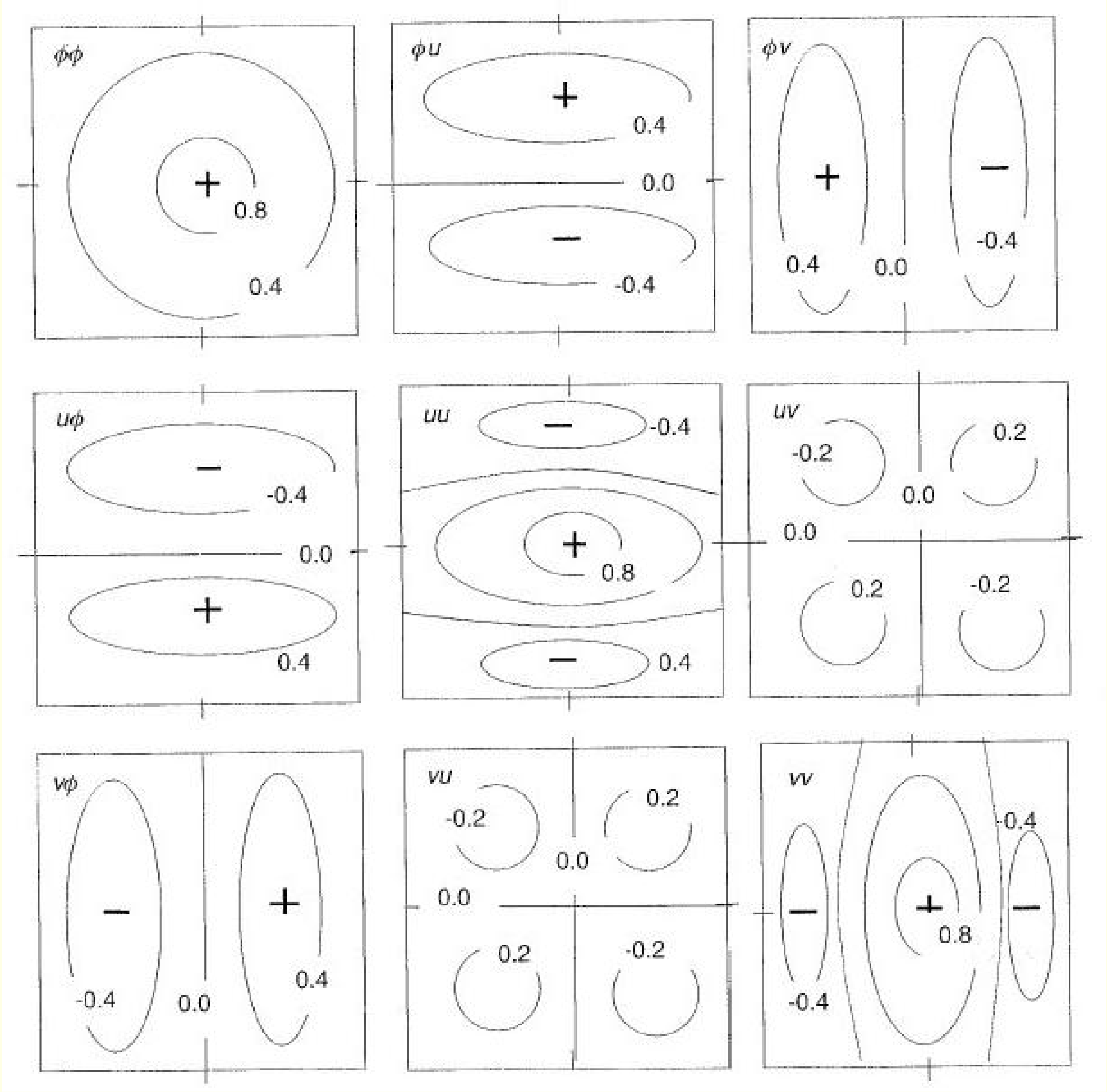
Typical "structure function" used in OI. The autocorelation of height is an isotropic Gaussian function. By geostrophy, the cross correlation with the tangential wind is maximum where the radial gradient of the height correlation is maximum.



Schematic illustration of the correlation of Φ-Φ.

The following figure shows schematically the shape of typical wind/height correlation functions used in OI.

Note that the u–h correlations have the opposite sign than the h–u correlations because the first and second variables correspond to the first and second points i and j respectively.



Correlation and cross-correlation functions.

J_B: The "NMC method"

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The model–forecast differences themselves provide a multivariate global forecast difference covariance.

This method has been shown to produce better results than previous estimates computed from *forecast minus observation* estimates.

Comparison of 3D-Var and OI

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- **3D-Var has allowed three-dimensional variational assimilation of radiances.**
- **The quality control of the observations becomes easier and more reliable when it is made in the space of the observations than in the space of the retrievals.**

End of §5.5