1 Summary

In the present work, a formulation of the sequential extended Kalman filter is proposed, based on recent findings concerning the deterministic dynamics of model error in deep contrast with previous approaches. This new scheme is applied in the context of the spatially distributed system proposed by Lorenz (1995). First it is found that, within 24 hours, the estimation error is accurately approximated by an evolution law in which the variance of the model error, assumed to be a deterministic process, evolves according to a quadratic law, while the correlation with the initial condition error appears to play only a minor role in the short time dynamics of the estimation error covariance. Second, the deterministic approach, incorporated into the classical extended Kalman filter equations, should be significantly improved if the filter accuracy as compared with the classical white noise assumption.

2 Error dynamics in the presence of initial condition and model errors

A general equation for the evolution of the error associated to the model and the true trajectories spans the same phase space. Model error is due to a-priori uncertainties in the specification of the parameters. Let the (unknown) true dynamics, the nature, and the model be represented in the form:

\[ \frac{dy(t)}{dt} = f(y(t), \lambda) \]

An equation for the evolution of the state estimation error \( \hat{y}(t) = y(t) - x(t) \) can be obtained by taking the difference between the above equations and exploiting up to the first order. After taking the expectation value the solution reads:

\[ \langle \hat{y}(t) \rangle = \int_0^t dt M(t-s) \langle \delta y(s) \delta y(0) \rangle \]

The model error is denoted \( \delta y = y - x \). Equation (1) states that, in the linear approximation, the error in the state estimate is given by the state estimated error covariance matrix in Eq. (7) can be approximated as:

\[ P(t) = M(t) + \langle \delta y(t) \delta y(0) \rangle \]

Remark - When white model error is bound to evolve quadratically, the correlation errors behave linearly with time and these terms may have a compensating effect resulting in a reduction of the total error.

In the case of a white noise model error, the short time evolution of the estimation error covariance matrix reads:

\[ P(t) = M(t) + \langle \delta y(t) \delta y(0) \rangle = \int_0^t dt M(t-s) \langle \delta y(s) \delta y(0) \rangle \]

Remark - By comparing Eqs. (6) and (5) we conclude that while the model error covariance matrix evolves linearly with time if the model error acts as a white noise source, it evolves quadratically in the case of deterministic model error.

Numerical Analysis

As a prototype of nonlinear chaotic dynamics, we use the Lorenz 36-variable model (Lorenz, 1995). The focus is on the investigation of the accuracy of the short time error dynamics, Eqs. (4) and (5).

Model error is simulated by perturbing its parameters with respect to the values \( \lambda = (\alpha, \beta, \gamma) = (1, 1, 28) \) used to represent the true dynamics.

3 EKF in the presence of model error

In the EKF, the forecast error covariance matrix, \( P^f \), is obtained by linearizing the model around its trajectory between two successive analyses, \( k \) and \( k+1 \), and assuming the forecast error bias is known and removed, \( \bar{P}(t) = P(t; k+1) \). The forecast error covariance matrix in the assimilation interval is given by:

\[ P^f = M(t) P(t; k+1) M(t)^T \]

Remark - As long as the observational forcing is frequent enough and the error is efficiently reduced by the assimilation of observations, the short time error dynamics, Eq. (6), should provide a reliable description of the actual model error evolution between two successive analyses. Similarly, Eq. (6) can be used in an EKF analysis cycle to estimate the state error covariance. This bias can then be removed from the forecast field before the latter is used in the analysis update.

White noise model error - If the model error is an additive white noise, its impact on the estimation error is related to the covariance of the random process, the matrix \( Q \). Consequently, assuming access to \( Q \) and an estimate of \( P(t) \), the model error covariance matrix in Eq. (7) can be estimated through the short time linear approximation:

\[ P(t) = P^f + Q(t) \]

Figure 1: Top: Mean error evolution for configurations \( \frac{F}{F} = +20\% \) (gray line and circles), \( \frac{F}{F} = −20\% \) (black line and squares), and \( \frac{F}{F} = 0 \) (dashed line). Bottom: Mean error evolution for different model errors, indicated in each panel. Actual error (continuous line); mean initial error condition - linear dynamics (dotted line); total error, model plus initial condition, approximated by Eq. (1) (dashed line).

Concluding remarks

- The analysis does not make use of any a-priori assumption on the model error dynamics except its deterministic character implying the existence of a short term universal behavior of model errors deduced in Nicolis (2003).
- The theoretical analysis is fully supported by numerical experiments in the context of a low order atmospheric model giving rise to chaotic behavior (Lorenz, 1995). It is shown that in the short time (less than 24 hours) the estimation error is accurately approximated by an evolution law in which the model error, treated as a deterministic process, is expanded in a Taylor series in time up to the first nontrivial order.
- The correlation between model and initial condition error has only a minor impact on the short time evolution of the error covariance matrix.

Deterministic model error - The short time evolution of the deterministic model error covariance is bound to be quadratic. In this case, by neglecting the correlation terms, the model error covariance matrix in Eq. (7) can be approximated as:

\[ P^f = P^f + Q(t) \]

In practice, the statistical information on the model error, \( \mu_p > 0 \) and \( Q(t) \) are not easy to evaluate. In the present work, since the origin of this model error is well known and the statistics over the whole attractor in order to get an invariant estimate of the model error, a single model error configuration is used.

Numerical Analysis

OSSEs with the EKF are performed in the context of the Lorenz 36-variable model (Lorenz, 1995). An homogeneous network of 108 observations is used at which model variables are recorded (\( i = 1, 3, 5, \ldots \)).

Figure 2: Top: Time running mean EKF analysis error variance. White noise case (left column) and deterministic process (right column). Parametric model error configurations: (i) (top panel) and (ii) (bottom panel). -20% EKF performance has been further examined by varying the assimilation intervals can be evaluated straightforwardly on the basis of the white noise hypothesis (left column), deterministic process (right column). Parametric model error configurations: (i) (top panel) and (ii) (bottom panel).