Model Error and Sequential Data Assimilation. A deterministic formulation.

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1 Summary

In the present work, a formulation of the sequential extended Kalman filter is proposed, based on recent findings concerning the deterministic dynamics of model error in deep contrast with previous approaches. This new scheme is applied in the context of the spatially distributed system proposed by Lorenz (1995). First it is found that, within 24 hours, the estimation error is accurately approximated by an evolution law in which the variance of the model error, assumed to be a deterministic process, evolves according to a quadratic law, while the correlation with the initial condition error appears to play only a minor role in the short time dynamics of the estimation error covariance. Second, the deterministic approach, incorporated into the classical extended Kalman filter equations, shows substantial improvements of the filter accuracy as compared with the classical white noise assumption.

2 Error dynamics in the presence of initial condition and model errors

A general equation for the evolution of the error associated to the estimate of the state of a system is derived, based on the use of an imperfect model of the system's dynamics, and an approximate knowledge of its initial state. We assume that the model and the true trajectories span the same phase space. Model error is due only to uncertainties in the specification of the parameters.

Let the (unknown) "true" dynamics, the *nature*, and the model be represented in the form:

$$\frac{d\mathbf{y}(t)}{dt} = \mathbf{f}(\mathbf{y}(t), \lambda), \qquad \frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, \lambda')$$

An equation for the evolution of the state estimation error $\delta\mathbf{x}(t) = \mathbf{y}(t) - \mathbf{x}(t)$ can be obtained by taking the difference between the above relations and expanding up to the first order. After taking the expectation value the solution reads:

$$< \delta\mathbf{x}(t) > \approx \mathbf{M}_{t,t_0} < \delta\mathbf{x}_0 > + \int_{t_0}^{t} d\tau \mathbf{M}_{t,\tau} < \delta\mu(\tau) > = < \delta\mathbf{x}^{ic} > + < \delta\mathbf{x}^{m} > \quad (1)$$

The model error is denoted $\delta\mu = \frac{\partial \mathbf{f}}{\partial \lambda}|_{\lambda'}\delta\lambda$. Equation (1) states that, in the linear approximation, the error in the state estimate is given by the sum of two terms, one relative to the evolution of initial condition error, $\delta\mathbf{x}^{ic}$, and another one relative to the model error, $\delta\mathbf{x}^{m}$.

Remark - Depending on the properties of model error, an initially unbiased estimate can evolve into a biased one. The important factor controlling the evolution of the mean state estimation error is the model error mean $< \delta\mu(t) >$.

The evolution equation of the state estimation error covariance matrix, assuming the estimation error bias is known and removed, reads:

$$\mathbf{P}(t) = < (\delta\mathbf{x}(t))(\delta\mathbf{x})^T(t) > \approx \mathbf{M}_{t,t_0} < (\delta\mathbf{x}_0)(\delta\mathbf{x})^T > \mathbf{M}^T_{t,t_0} + \int_{t_0}^{t} d\tau \int_{t_0}^{t} d\tau' \mathbf{M}_{t,\tau} < (\delta\mu(\tau))(\delta\mu(\tau'))^T) > \mathbf{M}^T_{t,\tau'} + \left[\mathbf{M}_{t,t_0} < (\delta\mathbf{x}_0)\left(\int_{t_0}^{t} d\tau \mathbf{M}_{t,\tau}\delta\mu(\tau)\right)^T >\right] + \left[\mathbf{M}_{t,t_0} < (\delta\mathbf{x}_0)\left(\int_{t_0}^{t} d\tau \mathbf{M}_{t,\tau}\delta\mu(\tau)\right)^T >\right]^T \quad (2)$$

The four terms of the r.h.s. of Eq. (2) depict the evolution of the initial condition error covariance, the model error covariance and their cross correlation matrices, respectively.

Remark - The net effect of the correlation between initial condition and model error may result in a reduction of the total estimation error.

For white noise Gaussian process $< \delta\mu >= 0$ and $< (\delta\mu(t))(\delta\mu(t'))^T >= \mathbf{Q}\delta(t - t')$, where $\mathbf{Q} = < (\delta\mu(t))(\delta\mu(t))^T >$ is a positive definite matrix representing the covariance of the process. In this hypothesis the evolution of estimation error covariance matrix reads:

$$\mathbf{P}_{wn}(t) = \mathbf{M}_{t,t_0} < (\delta\mathbf{x}_0)(\delta\mathbf{x}_0)^T > \mathbf{M}^T_{t,t_0} + \mathbf{P}^m_{wn}(t) = \mathbf{P}^{ic}(t) + \int_{t_0}^{t} \mathbf{M}_{t,\tau}\mathbf{Q}\mathbf{M}^T_{t,\tau}d\tau \quad (3)$$

with $\mathbf{P}^m_{wn}$ representing the model error covariance matrix in this white noise case.

Short time approximation

The short time approximation of the mean estimation error evolution becomes:

$$< \delta\mathbf{x}(t) > \approx \mathbf{M}_{t,t_0} < \delta\mathbf{x}_0 > + < \delta\mu_0 > (t - t_0) \quad (4)$$

As already noted by Nicolis (2003), the mean model error evolves linearly in time as long as the average $< \delta\mu_0 >$ is different from zero, otherwise the evolution is conditioned by higher orders of the Taylor expansion.

For the estimation error covariance matrix we get:

$$\mathbf{P}(t) \approx \mathbf{M}_{t,t_0} < (\delta\mathbf{x}_0)(\delta\mathbf{x}_0)^T > \mathbf{M}^T_{t,t_0} + [< (\delta\mu_0)(\delta\mathbf{x}_0)^T > \mathbf{M}^T_{t,t_0} + \mathbf{M}_{t,t_0} < (\delta\mathbf{x}_0)(\delta\mu_0)^T >](t - t_0) + < (\delta\mu_0)(\delta\mu_0)^T > (t - t_0)^2 \quad (5)$$

Remark - While model error is bound to evolve quadratically, the correlation errors behave linearly with time and these terms may have a compensating effect resulting in a reduction of the total error.

In the case of a white noise model error, the short time evolution of the estimation error covariance matrix reads:

$$\mathbf{P}(t) \approx \mathbf{M}_{t,t_0} < (\delta\mathbf{x}_0)(\delta\mathbf{x}_0)^T > \mathbf{M}^T_{t,t_0} + < (\delta\mu_0)(\delta\mu_0)^T > (t - t_0) \quad (6)$$

Remark - By comparing Eqs. (5) and (6) we conclude that while the model error covariance matrix evolves linearly with time if the model error acts as a white noise process, it evolves quadratically in the case of deterministic model error.

Numerical Analysis

As a prototype of nonlinear chaotic dynamics, we use the Lorenz 36-variable model (Lorenz, 1995). The focus is on the investigation of the accuracy of the short time error dynamics, Eqs. (4) and (5). The model evolution equations read:

$$\frac{dx_i}{dt} = \alpha(x_{i+1} - x_{i-2})x_{i-1} - \beta x_i + F = f(\mathbf{x}, \alpha, \beta, F), \qquad i = \{1, ..., 36\}$$

Model error is simulated by perturbing its parameters with respect to the values $\lambda = (\alpha, \beta, F) = (1, 1, 8)$ used to represent the true dynamics.

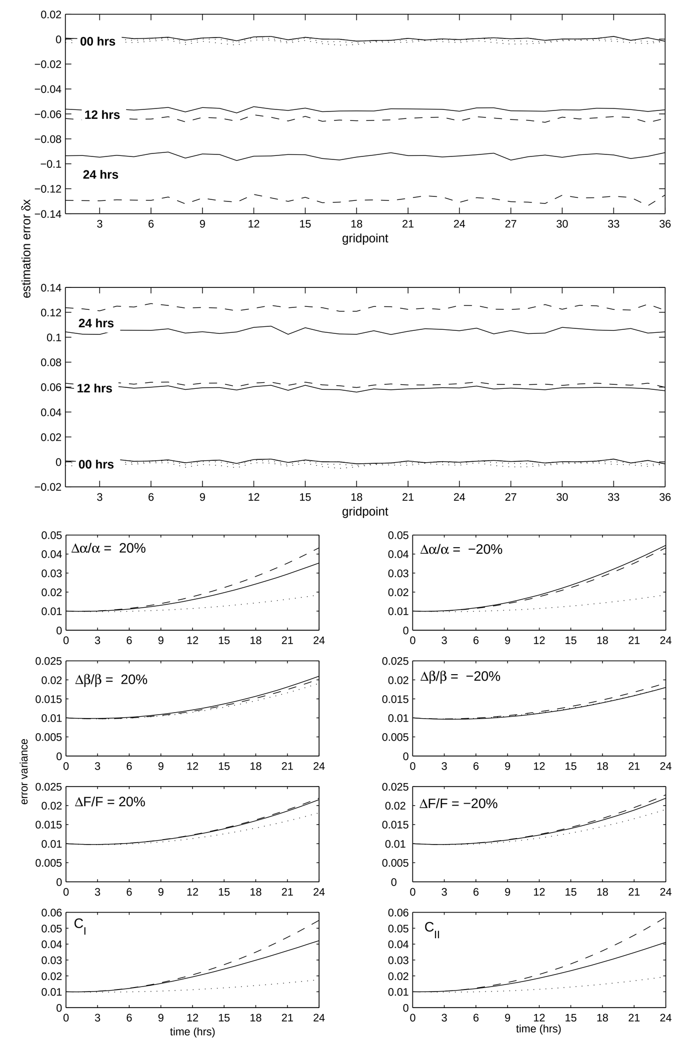

Figure 1: **TOP**: Mean error evolution for configurations C_I (C_I:$\frac{\Delta\alpha}{\alpha} = -20\%$, $\frac{\Delta\beta}{\beta} = -20\%$, $\frac{\Delta F}{F} = 20\%$; top panel) and C_{II} (C_{II}:$\frac{\Delta\alpha}{\alpha} = 20$, $\frac{\Delta\beta}{\beta} = 20\%$, $\frac{\Delta F}{F} = -20\%$; bottom panel). Actual error (continuous line); mean initial condition error - linear dynamics (dotted line); total error, model plus initial condition, approximated by Eq. (4) (dashed line). **BOTTOM**: Mean square error evolution for different model errors, indicated in each panels. Actual error (continuous line); initial condition error - linear dynamics (dotted line); total error approximated by Eq. (5) (dashed line).

Concluding remarks

- The analysis does not make use of any a-priori assumption on the model error dynamics except its deterministic character implying the existence of a short term universal behavior of model errors deduced in Nicolis (2003).
- The theoretical analysis is fully supported by numerical experiments in the context of a low order atmospheric model giving rise to chaotic behavior (Lorenz, 1995). It is shown that in the short time (less than 24 hours) the estimation error is accurately approximated by an evolution law in which the model error, treated as a deterministic process, is expanded in a Taylor series in time up to the first nonlinear order.
- The correlation between model and initial condition error has only a minor impact on the short time evolution of the error covariance matrix.

3 EKF in the presence of model error

In the EKF, the forecast error covariance matrix, $\mathbf{P}^f$, is obtained by linearizing the model around its trajectory between two successive analysis times t_k and t_{k+1}. Assuming that the model error is uncorrelated with the analysis error, the evolution equation of the forecast error covariance matrix within the assimilation interval is given by:

$$\mathbf{P}^f_{k+1} = \mathbf{M}_{k+1,k}\mathbf{P}^a_k\mathbf{M}^T_{k+1,k} + \mathbf{P}^m_{k+1}, \quad (7)$$

Remark - As long as the observational forcing is frequent enough and the error is efficiently reduced by the assimilation of observations, the short time error dynamics, Eq. (5), should provide a reliable description of the actual model error evolution between two successive analyses. Similarly, Eq. (4) can be used in an EKF analysis cycle to estimate the bias in the forecast error; this bias can then be removed from the forecast field before the latter is used in the analysis update.

White noise model error - If the model error is an additive white noise, its impact on the estimation error is related to the covariance of the random process, the matrix $\mathbf{Q}$. Consequently, assuming access to (or to an estimate of) $\mathbf{Q}$, the model error covariance matrix in Eq. (7) can be estimated through the short time linear approximation:

$$\mathbf{P}^m = \mathbf{P}^m_{wn} \approx < (\delta\mu - < \delta\mu >)(\delta\mu - < \delta\mu >)^T > \tau = \mathbf{Q}\tau \quad (8)$$

$\tau = t_{k+1} - t_k$ being the assimilation interval.

Deterministic model error - The short time evolution of the deterministic model error covariance is bound to be quadratic. In this case, by neglecting the correlation terms, the model error covariance matrix in Eq. (7) can be approximated as:

$$\mathbf{P}^m = \mathbf{P}^m_{dp} \approx < (\delta\mu - < \delta\mu >)(\delta\mu - < \delta\mu >)^T > \tau^2 = \mathbf{Q}\tau^2 \quad (9)$$

In practice, the statistical information on the model error, $< \delta\mu >$ and $\mathbf{Q}$, are not easy to evaluate. In the present work, since the origin of this model error is known, we perform the statistics over the whole attractor in order to get an invariant estimate of the model error moments.

Numerical Analysis

OSSEs with the EKF are performed in the context of the Lorenz 36-variable (Lorenz, 1995). An homogeneous network of 18 noisy observations is used at which model variables are recorded (x_i, $i = 1, 3, ..., 35$).

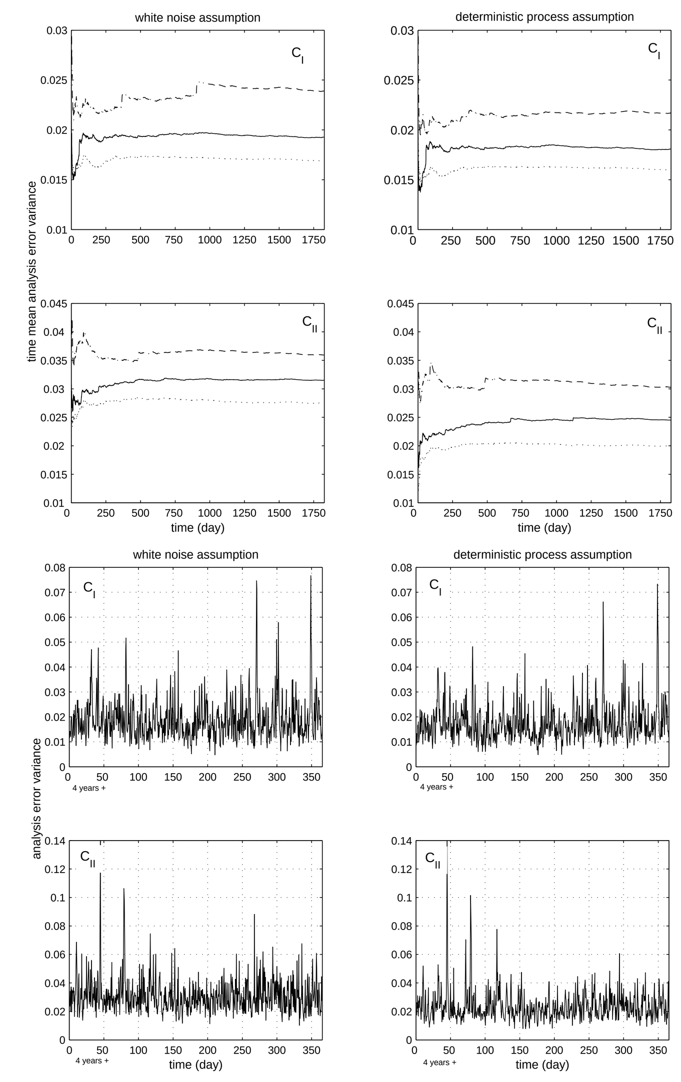

Figure 2: **TOP**: Time running mean EKF analysis error variance. White noise case (left column) and deterministic process (right column). Parametric model error configurations: C_I (top panels) and C_{II} (bottom panels); $\tau =$ 12 hours (dash-dotted lines), 6 hours (continuous lines) and 3 hours (dashed lines). **BOTTOM**: EKF analysis error variance as a function of time for the experiment with $\tau =$6 hours, during the last year of the simulation. White noise hypothesis (left column), deterministic process (right column). Parametric model error configurations: C_I (top panels) and C_{II} (bottom panels).

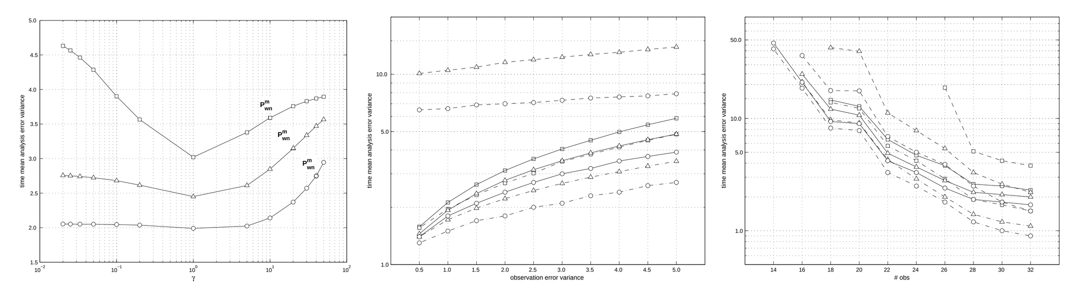

Figure 3: **LEFT**: EKF time average analysis error variance as a function of the coefficient γ, $\mathbf{P}^m = \gamma\mathbf{P}^m_{dp}$. The parametric model error configuration is C_{II} and $\tau = 3$ hours (circles), $\tau = 6$ hours (triangles) and $\tau = 12$ (squares). **CENTER**: EKF time average analysis error variance as a function of the observation error variance (expressed as a percentage of the system's climate variance) for parametric model error configuration C_{II}; $\tau = 3$ hours (circles), $\tau = 6$ hours (triangles) and $\tau = 12$ (squares). The different lines refer to the perfect model assumption (dashed lines), the white noise (continuous lines) and the deterministic process (dash-dotted lines). **RIGHT**: EKF time average analysis error variance as a function of the number of observations for the parametric model error configuration C_{II}; same line and marks notation as in the center panel.

Concluding remarks

- The possibility of using the deterministic approach to account for the model error dynamics in the extended Kalman filter has been explored and compared to the classical white noise approach.
- The numerical analysis gives a clear indication of the substantial improvement of the filter accuracy when model error is assumed to act as a deterministic process.
- The EKF performance has been further examined by varying either the assimilation intervals or the observational network properties. As long as the assimilation interval is short enough the filter employing the quadratic model error evolution law (deterministic process assumption) outperforms systematically the white noise case.
- The existence of this universal, short time, quadratic law for the evolution of model error covariance might turn to be useful in all the situations in which access to statistical information on the parametric model error is available and can be used to estimate the model error covariance matrix.
- If, for a given assimilation interval, an optimal model error covariance matrix is at hand, the optimal matrix at different assimilation intervals can be evaluated straightforwardly on the basis of the quadratic law.

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Key References

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